Last time | Lagrangian for the Electromagnetic Field and Maxwell Equations (cont'd)

Four-Vector of Electromagnetic Current (cont'd)

\[ \text{Def.} \quad \text{charge density} \quad j^v(x,t) = \frac{\text{charge}}{\text{volume}} \]

For a discrete set of point charges \( q_i \), \( i=1, \ldots, n \), we get

\[ j^v(x,t) = \sum_{i=1}^{n} q_i \delta^3(x - \vec{x}_i(t)) \]

\[ \text{Def.} \quad \delta \text{-function:} \]

\[ (i) \quad \delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases} \]

\[ (ii) \quad \int_{-\infty}^{\infty} dx \ f(x) \delta(x-a) = f(a). \]

\[ \delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi} \varepsilon} e^{-x^2/\varepsilon^2} \]

\[ \nu \text{ can be thought of as a limit of a smooth function} \]

Properties of \( \delta \)-function:

1. \( \delta(x) = \delta(-x) \)

2. \( \int_{-\infty}^{\infty} dx \ f(x) \delta^{(n)}(x-a) = (-1)^n f^{(n)}(a) \)
(3) \[ S(f(x)) = \sum_{i=1}^{n} \frac{1}{|f'(x_i)|} S(x-x_i) \], where \( x_i \) are roots of \( f(x) \), such that \( f(x_i) = 0 \).

(4) \[ S(x-y) = S(x^1-y^1) S(x^2-y^2) S(x^3-y^3) \]

Also,

\[ S'(x-y) = S(x^0-y^0) S(x^1-y^1) S(x^2-y^2) S(x^3-y^3) \]

In a 4-dimensional S-function, \( x^i \) and \( y^i \) are 4-vectors.
To prove (ii) for a more general class of functions go to Fourier transform

\[ f(x) = \int \frac{dk}{2\pi} \ e^{-ik\cdot x} \ \tilde{f}(k) \]

\& we will have to prove (ii) only for exponents

\[ f(x) \sim e^{-ik\cdot x} \]

Properties of delta functions:

1. \( \delta(-x) = \delta(x) \) (it's an even function)

2. \( \int_{-\infty}^{\infty} dx \ f(x) \ \delta^{(n)}(x-a) = (-1)^n f^{(n)}(a) \)

in particular

\[ \int_{-\infty}^{\infty} dx \ f(x) \ \delta'(x-a) = -f'(a) \]

Proof: \( \int_{-\infty}^{\infty} dx \ f(x) \ \frac{d}{dx} \delta(x-a) = \left. f(x) \delta(x-a) \right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx \ f'(x) \ \delta(x-a) = -f'(a) \)

3. \( \delta(f(x)) = \sum_{i=1}^{n} \frac{1}{|f'(x_i)|} \delta(x-x_i) \)

where \( x_i, i=1, \ldots, n \) are roots of \( f(x) \), \( f(x_i) = 0 \).
Proof: \[
1 = \int d\ell f(x) \delta(f(x)) = \int dx \lfloor f'(x) \rfloor \delta(f(x)) \quad (42)
\]
Integrate near one of the roots \( x_i \).

( need abs value \( |f'(x)| \) to have the right direction of the integral over \( x \), \( \int_{x_{-\Delta}}^{x_{+\Delta}} \) and not \( \int_{x_{-\Delta}} \) )

\[ \Rightarrow \text{we see that} \quad \delta(x-x_i) = \frac{1}{|f'(x_i)|} \delta(f(x)) \]

for \( x \) near \( x_i \) \[ \Rightarrow \delta(f(x)) = \frac{1}{|f'(x_i)|} \delta(x-x_i) \]

in the vicinity of \( x_i \) \[ \Rightarrow \delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|} \delta(x-x_i) \]

after summing over all roots.

(4) \[ S^3(x-y) = \delta(x_1-y_1) \delta(x_2-y_2) \delta(x_3-y_3) \]

(can treat this as a definition of \( \delta \)-func.)

Also \[ S^7(x-y) = S(x^0-y^0) S(x^1-y^1) S(x^2-y^2) S(x^3-y^3) \]

(9-dim \( \delta \)-function, \( x^m \) & \( y^m \) are 4-vectors)
(Def.) Current density \( \vec{J}(\vec{x}, t) \) is defined as

the current per unit area or

\[
\vec{J} = \frac{\text{charge} \cdot \text{velocity}}{\text{volume}}
\]

For point charges write

\[
\vec{J}(\vec{x}, t) = \sum_{i=1}^{n} q_i \vec{V}_i(t) \cdot S^3(\vec{x} - \vec{x}_i(t))
\]

Charge conservation:

Imagine a volume \( V \): the change in total charge inside the volume is equal to the amount of charge that flowed in/out the volume:

\[
\Delta Q = \int_V d^3x \left[ \rho(\vec{x}, t + \Delta t) - \rho(\vec{x}, t) \right] = \Delta t \int_S d\vec{a} \ \hat{n} \cdot \vec{J}
\]

where \( \hat{n} \) is a unit normal to the surface vector pointing outward, \( d\vec{a} \) a surface element.

Divergence Theorem: For a vector field \( \vec{V}(\vec{x}) \) we have

\[
\int_V d^3x \ \nabla \cdot \vec{V} = \int_S d\vec{a} \ \hat{n} \cdot \vec{V}
\]

(see Arfken Sec. 3.8)
Using the divergence theorem we write
\[ \oint_S d\alpha \hat{n} \cdot \vec{J} = \int d^3x \nabla \cdot \vec{J} \]
such that
\[ \int d^3x \left[ \rho(x, t+\Delta t) - \rho(x, t) \right] = -\int d^3x \nabla \cdot \vec{J} \cdot \Delta t \]
\[ = \int d^3x \rho(x, t+\Delta t) - \rho(x, t) \Delta t \frac{\partial \rho(x, t)}{\partial t} = -\Delta t \nabla \cdot \vec{J} \]
\[ \Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \]
continuity equation manifests charge conservation.

As we know \( x^\mu = (ct, \vec{x}) \) and \( \partial_\mu = \frac{\partial}{\partial x^\mu} \) are 4-vectors \( \Rightarrow \) defining an object
\[ J^\mu = (\rho, \vec{J}) \]
we rewrite the continuity equation as
\[ \partial_\mu J^\mu = 0 \]
This is true in any frame \( \Rightarrow \) a Lorentz-invariant statement. As \( \partial_\mu J^\mu \) is Lorentz-invariant, and \( \partial_\mu \) is a 4-vector \( \Rightarrow J^\mu \) is a 4-vector too!
\[
\Rightarrow \begin{array}{c}
\mathcal{J}^\mu \\
\partial_\mu \mathcal{J}^\mu = 0
\end{array}
\]
is a 4-vector of current

is often referred to as the current conservation

The action for charge-field interactions is

\[
\mathcal{S}^i_{\text{int}} = -\frac{q_i}{e} \int dt \frac{1}{\lambda} u_\mu A^\mu = \Rightarrow \text{for the set of } n \text{ point charges at hand write}
\]

\[
\mathcal{S}^i_{\text{int}} = -\frac{1}{e} \int dt d^3x \sum_i g_i \frac{1}{\delta_i} \mathcal{J}_\mu \delta^3(x - x_i) A^\mu(x)
\]

Since

\[
\begin{align*}
\mathcal{J}_\mu &= \sum_i g_i \vec{v}_i \delta^3(x - x_i) \\
\Rightarrow \mathcal{J}_\mu &= \sum_i g_i \vec{v}_i \delta^3(x - x_i)
\end{align*}
\]

\[
\Rightarrow \mathcal{J}_\mu = (c, \vec{v}) = \sum_i g_i \frac{1}{\delta} u_\mu \delta^3(x - x_i)
\]

\[
\Rightarrow \mathcal{S}^i_{\text{int}} = -\frac{1}{e} \int dt d^3x \mathcal{J}_\mu A^\mu
\]

Define a 4-dim integration measure

\[
d^4x = dx^0 dx^1 dx^2 dx^3 = c dt d^3x
\]
Note that $d^4x$ is Lorentz-invariant.

Under Lorentz transformation $x'^{\mu} = \Lambda^{\mu}_{\nu} x^\nu$, we get $\, d^4x' = (\det \Lambda) \, d^4x$, but $\det \Lambda = +1$.

$\Rightarrow \, d^4x' = d^4x \quad \text{(Lorentz-invariant)}$

E.g. $\det \begin{pmatrix} \gamma & -\beta \gamma & 0 & 0 \\ -\beta \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \gamma^2 (1-\beta^2) = 1 \quad \text{(true for all } \Lambda)\text{)}$

$\Rightarrow \, \text{Using } d^4x \text{ write}$

\[ S_{\text{int}} = -\frac{1}{c^2} \int d^4x \, J_m A^m \]

In general, $S_{\text{int}} = \int d^4x \, L_{\text{int}}$, where $L$ is called the Lagrangian density (such that $L = \int d^3x \, \xi$).

We get \[ L_{\text{int}} = -\frac{1}{c^2} \, J_\mu A^\mu \]

Note that $S' = \int d^4x \, \xi \Rightarrow L$ is Lorentz-invariant in general.