Last time

Def. Interval: $s_{12}^2 = c^2 (t_2 - t_1)^2 - |\vec{x}_1 - \vec{x}_2|^2$.

Infinite interval: $ds^2 = c^2 dt^2 - d\vec{x}^2$.

Def. Proper time: $d\tau = \frac{ds}{c}$

$\Rightarrow \tau_2 - \tau_1 = \int_{\tau_1}^{\tau_2} dt \sqrt{1 - \beta^2(t)} \Rightarrow \Delta \tau \approx \Delta t$

Lorentz contraction: $l = l_0 \sqrt{1 - \frac{v^2}{c^2}}$

$t$ proper length (length in rest frame)

Velocity Transformations:

\[
\begin{align*}
V_x &= \frac{V_x' + u}{1 + \frac{uv_x'}{c^2}} \\
V_y &= \frac{\sqrt{1 - \frac{u^2}{c^2}}}{1 + \frac{uv_x'}{c^2}} \\
V_z &= \frac{V_z'}{1 + \frac{uv_x'}{c^2}}
\end{align*}
\]

\[
\tan \theta = \frac{v' \sin \theta'}{\sqrt{v'(\cos \theta' + u)}}
\]
Four - vectors.

We have seen one example: $x^0 = ct$, $x'^1 = x$, $x'^2 = y$, $x'^3 = z$

$$
\begin{pmatrix}
    x'^0 \\
    x'^1 \\
    x'^2 \\
    x'^3
\end{pmatrix} = \begin{pmatrix}
    1 & -\beta & 0 & 0 \\
    \beta & \gamma & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
    x^0 \\
    x^1 \\
    x^2 \\
    x^3
\end{pmatrix} \Rightarrow x'^{\mu} = \Lambda^{\mu}_{\nu} x^\nu. \\
\frac{\partial x'^{\mu}}{\partial x^0} = \Lambda^\mu_0.
$$

A general Lorentz transformation can be

- a boost along $x$, $y$, or $z$ axis (a velocity, $\beta$)
- a rotation around the $x$, $y$, or $z$ axis (an angle $\theta$)
A 4-vector $A^\mu$ is a set of 4 quantities $(A^0, A^1, A^2, A^3)$, which under Lorentz transformations transform as

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$$

(just like

$$A'^\mu = \delta^\mu_\nu A^\nu$$

$\mu, \nu = 0, 1, 2, 3$

summation over $\nu$

is implied.

$\Rightarrow A^\mu, \mu = 0, 1, 2, 3$ is a contravariant vector if it transforms according to:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$$

(equivalent to above)

$\Rightarrow B_\mu, \mu = 0, \ldots, 3$ is a covariant vector if

$$B'_\mu = \frac{\partial x'^\mu}{\partial x^\nu} B^\nu$$

Example: $\frac{\partial \psi}{\partial x^\mu}$ is a covariant vector as $\frac{\partial \psi}{\partial x^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \psi}{\partial x^\nu}$.

One can define tensors by

$$A'^\mu B'^\nu = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} A^\alpha B^\beta \Rightarrow \text{rank two contravariant tensor would be } C'^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} C^\alpha_\beta, \text{ etc.}$$

In general $T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_m}$ transforms as $A_{\mu_1} \ldots A_{\mu_n} B^{\nu_1} \ldots B^{\nu_m}$. 
**Definition**  Scalar (inner) product of 2 vectors is defined by  \( A_\mu \cdot B^\mu \) (summation assumed)

Let's prove that it's Lorentz invariant:

\[
A'_\mu \cdot B'^\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu \frac{\partial x'^\mu}{\partial x^\beta} B^\beta = \frac{\partial x^\nu}{\partial x^\beta} A_\nu B^\beta = S_{\alpha\beta},
\]

\[
A_\lambda B^\lambda = A_\alpha B^\alpha. \  \text{Q.E.D.}
\]

The interval is a scalar: (it's Lorentz invariant)

\[
ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2
\]

**Define** the metric tensor by

\[
ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu
\]

\[
g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\]

Note that  \( dx_\mu \, dx^\mu \) is also a Lorentz-scalar.

Identifying  \( dx_\mu = g_{\mu\nu} \, dx^\nu \) we see that  \( g_{\mu\nu} \) lowers indices of 4-vectors, tensors, etc.

**Example**  \( x^\mu = (ct, \vec{x}) \),  \( x_\mu = g_{\mu\nu} \, x^\nu = (ct, -\vec{x}) \).
\[
A_\mu = g_{\mu \nu} A^\nu \Rightarrow A^\mu = g^{\mu \nu} A_\nu
\]

where
\[
S^\nu = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

\[
S^\nu = g^{\mu \nu} S_\mu = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

indeed, as
\[
A_\mu B^\mu = S^\nu A_\mu B_\nu = g^{\mu \nu} A_\mu B_\nu
\]

**Define** an abbreviated notation: \( \Theta_\mu \equiv \frac{\partial}{\partial x^\mu} \)

\[
\Theta_\mu = \frac{\partial}{\partial x^\mu} 
\]

\( \Theta_\mu \) is a **covariant vector**

\( \Theta^\mu \) is a **contravariant vector** (check!)

\[
\Theta_\mu A^\mu
\]

Lorentz invariant

4d Laplace operator \( \nabla^2 = \frac{\partial^2}{\partial x^0 \partial x^0} + \frac{\partial^2}{\partial x^1 \partial x^1} + \frac{\partial^2}{\partial x^2 \partial x^2} + \frac{\partial^2}{\partial x^3 \partial x^3} \)

also Lorentz invariant. (can you prove this?)

**4 - Velocity**

Let's define a 4-vector for velocity:

\[
dx^\mu = (dx^0, dx^1, dx^2, dx^3) \Rightarrow V^\mu = \frac{dx^\mu}{dt}
\]