Renormalization of the Drude Conductivity by the Electron-Phonon Interaction

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It is commonly believed that the Drude conductivity is not renormalized by many-body interactions. To the contrary, we show that for impure metals the Drude conductivity is decreased by a temperature-dependent factor proportional to the electron-phonon coupling λ. We demonstrate this effect by both the linear response and the quantum transport approaches. This effect, while comparable to the bare Drude conductivity, will be masked by the large temperature dependence of electron-phonon scattering. We expect such many-body corrections to other transport coefficients.

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Three decades ago Prange and Kadanoff [1] studied which electron transport coefficients were renormalized by the electron-phonon interaction. They concluded that the stationary quantum transport equation for electrons interacting with phonons is accurately described by the usual Boltzmann equation without any renormalization factors. Therefore such coefficients as dc electrical conductivity, thermopower, and thermal conductivity would not be affected by the electron-phonon interaction.

Up to now their conclusions have been disputed only for the thermopower. Initially Opsal, Thaler, and Bass [2] found experimentally the renormalization correction with the same temperature dependence as the electron-phonon mass enhancement factor $1 + \lambda(T)$, where $\lambda(T)$ is the electron-phonon coupling. This result attracted many theoretical works [3]. Recent investigations [4,5] show that the quantum interference between electron-phonon and electron-impurity scattering plays a crucial role in the thermopower renormalization. In particular, the sign and magnitude of the renormalization factor depend strongly on the sign and magnitude of the electron-impurity potential, respectively. Except for the thermopower, the strong current opinion (for example, see Refs. [6] and [7]) is that all other electron transport coefficients are not renormalized by the electron-phonon interaction. This is not true.

In this paper we show that if $\omega_D\tau > 1$ (ω_D is the Debye frequency and τ is the electron momentum relaxation rate due to electron-impurity scattering), the phonon renormalization of the Drude conductivity σ_0, calculated in the first order in λ, is

$$\sigma = \sigma_0\left\{1 - 2\lambda\left[1 - \left(\frac{q_D}{2p_F}\right)^2\right]\right\}. \tag{1}$$

In Eq. (1), p_F is the Fermi momentum, and q_D is the Debye wave vector; λ is defined as $-d\Sigma_{e\text{-ph}}/d\epsilon$, where $\Sigma_{e\text{-ph}}$ is the electron self-energy due to the electron-phonon interaction (for details see p. 176 of Ref. [8]). When $\omega_D\tau < 1$, the renormalization effect drops to zero as $\omega_D\tau$.

For arbitrary interaction, such as the electron-electron, electron-paramagnon, electron-magnon interaction, our result [Eq. (1)] may be extended as follows. If the characteristic energy scale (ω_0) of the corresponding electron self-energy (Σ) is larger than the electron-impurity scattering rate ($1/\tau$), the renormalization of the Drude conductivity is described by Eq. (1), where $\lambda = -d\Sigma/d\epsilon$. At $\omega_0\tau < 1$ the renormalization effect is negligible.

We stress that, on the one hand, our results lie outside the scope of the results by Prange and Kadanoff. They discussed the renormalization of the Bloch-Gruneisen term, or, in other words, the phonon renormalization of the electron-phonon scattering processes. We discuss *the phonon renormalization of the Drude conductivity* due to electron-impurity scattering. On the other hand, we show that terms responsible for the conductivity renormalization in our case were not considered in the original Prange and Kadanoff paper.

Linear response approach.—For linear response calculations either the Matsubara technique or the Keldysh diagrammatic technique can be used. We choose the Keldysh description, because it is ideal for the quantum transport equation, which will be considered in the next.

The electron and phonon subsystem are described by advanced, retarded, and kinetic (Keldysh) Green functions. The retarded and advanced electron Green functions averaged over impurity positions are

$$G_0^R(\mathbf{p},\epsilon) = [G^A(\mathbf{p},\epsilon)]^* = \left(\epsilon - \xi_p + \frac{i}{2\tau}\right)^{-1}, \qquad \xi_p = (p^2 - p_F^2)/2m\,. \tag{2}$$

The phonon Green functions are

$$D^R(\mathbf{q},\omega) = [D^A(\mathbf{q},\omega)]^* = (\omega - \omega_\mathbf{q} + i0)^{-1} + (\omega + \omega_\mathbf{q} + i0)^{-1}. \tag{3}$$

The kinetic Green functions in the thermodynamic equilibrium may be written as

$$G_0^C(\mathbf{p},\epsilon) = S_0(\epsilon)\,[G_0^A(\mathbf{p},\epsilon) - G_0^R(\mathbf{p},\epsilon)], \qquad S_0(\epsilon) = -\tanh\left(\frac{\epsilon}{2T}\right), \tag{4}$$

$$D^C(\mathbf{q},\omega) = [2N(\omega)+1][D^R(\mathbf{q},\omega) - D^A(\mathbf{q},\omega)],$$
$$2N(\omega)+1 = \coth\left(\frac{\omega}{2T}\right). \tag{5}$$

In the diagrammatic technique the conductivity is related to the retarded electronic loop with two vector vertices $e\mathbf{v}\cdot\mathbf{n}$, where e is the electron charge, $\mathbf{v}$ is the electron velocity corresponding to the Green function forming the vertex, and $\mathbf{n}$ is a unit vector. All important diagrams are shown in Fig. 1. Under the conditions $p_F l \gg 1$, $q_D l \gg 1$ we can neglect other diagrams (not shown in Fig. 1) with additional electron-impurity interaction and inelastic electron-impurity scattering.

Studying the renormalization effect one takes into account only terms proportional to the real part of the phonon Green function. The analytical expression for the first diagram is, with $P = (\mathbf{p},\epsilon)$, $Q = (\mathbf{q},\omega)$,

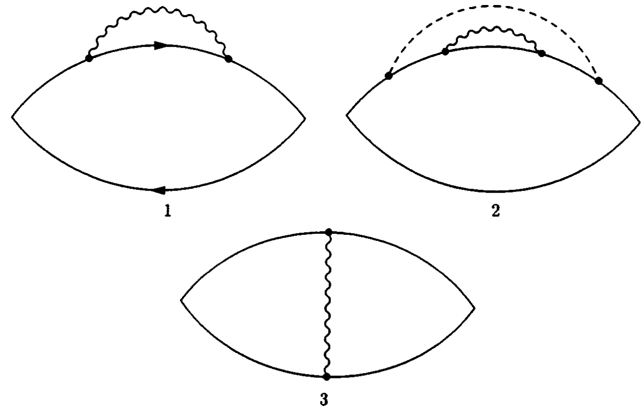

FIG. 1. Diagrams for the conductivity renormalization to the first order in the electron-phonon interaction. The first and second diagrams cancel exactly for the conductivity.

$$\sigma_1 = 2e^2\int \frac{d^4P}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}|g_q|^2(\mathbf{v}\cdot\mathbf{n})^2\tau S_0(\epsilon+\omega)\frac{\partial S_0(\epsilon)}{\partial\epsilon}\mathrm{Im}[G_0^A(\mathbf{p},\epsilon)]^2\mathrm{Im}G_0^A(\mathbf{p}+\mathbf{q},\epsilon+\omega)\mathrm{Re}D(\mathbf{q},\omega), \tag{6}$$

where g_q is the electron-phonon matrix element. The second diagram may be written $\sigma_2 = -\sigma_1 W$, where

$$W = \frac{1}{\pi\nu\tau}\int\frac{d\mathbf{p}'}{(2\pi)^3}G_0^A(\mathbf{p}',\epsilon)G_0^R(\mathbf{p}',\epsilon) = 1\,. \tag{7}$$

Therefore the contributions of the first two diagrams mutually cancel. The third diagram is

$$\begin{aligned}\sigma_3 = 2e^2\int &\frac{d^4P}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}|g_q|^2(\mathbf{v}\cdot\mathbf{n})\left(\mathbf{v}\cdot\mathbf{n}+\frac{\mathbf{q}\cdot\mathbf{n}}{m}\right)\tau\mathrm{Re}D(\mathbf{q},\omega)\\ &\times\Bigg[S_0(\epsilon)\frac{\partial S_0(\epsilon+\omega)}{\partial\epsilon}\mathrm{Im}[G_0^A(\mathbf{p},\epsilon)]^2\mathrm{Im}G_0^A(\mathbf{p}+\mathbf{q},\epsilon+\omega)\\ &+S_0(\epsilon+\omega)\frac{\partial S_0(\epsilon)}{\partial\epsilon}\mathrm{Im}G_0^A(\mathbf{p},\epsilon)\mathrm{Im}[G_0^A(\mathbf{p}+\mathbf{q},\epsilon+\omega)]^2\Bigg].\end{aligned} \tag{8}$$

Changing variables $\mathbf{p}\to\mathbf{p}+\mathbf{q}$ and $\epsilon\to\epsilon+\omega$ and then $\mathbf{q}\to-\mathbf{q}$ and $\omega\to-\omega$, we may show that the first and second terms in the large square brackets of Eq. (8) are equal. Calculating the first term at low temperatures $T<\omega_D, 1/\tau$, we note that

$$\int\frac{d^4Q}{(2\pi)^4}|g_q|^2\frac{\partial S_0(\epsilon+\omega)}{\partial\epsilon}\times \mathrm{Im}G_0^A(\mathbf{p}+\mathbf{q},\epsilon+\omega)\mathrm{Re}D(\mathbf{q},\omega) = -\frac{\partial\Sigma_{e\text{-ph}}}{\partial\epsilon}, \tag{9}$$

$$\int\frac{d\epsilon}{2\pi}d\xi_p\,S_0(\epsilon)\mathrm{Im}[G_0^A(\mathbf{p},\epsilon)]^2 = -1\,. \tag{10}$$

In Eq. (10) the integral is dominated by region $\epsilon,\xi_p\sim\tau^{-1}$. Therefore, if $\omega_D>\tau^{-1}$, in Eqs. (9) and (8) we can set $-\partial\Sigma_{e\text{-ph}}/\partial\epsilon=\lambda$, independent of ϵ. After that the angular integration of the factor $(\mathbf{v}\cdot\mathbf{n})^2$ yields $-\lambda\sigma_0$. To perform the angular integration of the term $(\mathbf{v}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n})/m$ it is necessary to include it in the left hand side of Eq. (9) to yield the multiplication factor $-(q_D/2p_F)^2$. Finally the contribution of the third diagram gives the renormalization correction displayed in Eq. (1). If $\omega_D\tau<1$, the integration region in Eq. (8) is limited by ω_D, and therefore the renormalization effect drops to zero as $\omega_D\tau$.

Quantum transport equation.—The conductivity renormalization is one of many quantum interference phenomena in normal metals for which one should use the rigorous approach capable of accounting for the quantum nonlocal corrections in the form of the Poisson brackets [4,9,10,11]. The linearized quantum transport equation for the distribution function S of electrons scattering from impurities and phonons has the form

$$e(\mathbf{v}\cdot\mathbf{E})\frac{\partial S_0}{\partial\epsilon} = I_{e\text{-imp}}(S) + I_{e\text{-ph}}(S)\,, \tag{11}$$

where $\mathbf{E}$ is the electric field and the oscillation integrals $I_{e\text{-imp}}$ and $I_{e\text{-ph}}$ can be expressed in terms of the corresponding self-energies by the equation [4,6,9]

$$I(S) = -i[\Sigma^C - S(\Sigma^A-\Sigma^R)] + \tfrac{1}{2}\{\Sigma^A+\Sigma^R, S_0\}, \tag{12}$$

where the Poisson bracket correction is

$$\{A,B\} = e\mathbf{E}\left(\frac{\partial A}{\partial\epsilon}\frac{\partial B}{\partial\mathbf{p}} - \frac{\partial B}{\partial\epsilon}\frac{\partial A}{\partial\mathbf{p}}\right). \tag{13}$$

The dominance of electron-impurity scattering in the momentum relaxation permits the solution of the transport equation [Eq. (11)] by iteration: $S = S_0 + \phi_0 + \phi_1$, where ϕ_0 is the nonequilibrium correction determined by electron-impurity scattering

$$\phi_0(\mathbf{p}, \epsilon) = -\tau e(\mathbf{v} \cdot E) \frac{\partial S_0(\epsilon)}{\partial \epsilon}. \tag{14}$$

The correction ϕ_1 includes the effects of the electron-phonon interaction

$$\phi_1(\mathbf{p}, \epsilon) = \tau I_{e\text{-ph}}. \tag{15}$$

In the quantum transport equation method, the corrections to the electric current (conductivity) may originate from the correction to the distribution function as well as from various corrections to $\mathrm{Im}G^A$, which can be treated as corrections to the electron density of states. Our calculations show that the third diagram in the linear response method is related to the following two terms in the quantum transport equation:

$$\delta\mathbf{J}_e = \delta\sigma\mathbf{E} = 2\int \frac{d^4P}{(2\pi)^4}\mathbf{v}[S_0(\epsilon)\mathrm{Im}\delta_{\mathrm{ph}}G^A(\phi_0) + \phi_1(\mathbf{p}, \epsilon)\mathrm{Im}G_0^A(\mathbf{p}, \epsilon)]. \tag{16}$$

The first term in Eq. (8) corresponds to the nonequilibrium correction to $\mathrm{Im}G^A$ due to ϕ_0,

$$\begin{aligned}\delta_{\mathrm{ph}}G^A(\phi_0) &= (G_0^A)^2\Sigma^A_{e\text{-ph}}(\phi_0)\\ &= [G_0^A(\mathbf{p}, \epsilon)]^2 \int \frac{d^4Q}{(2\pi)^4}|g_q|^2\tau e\left(\mathbf{v} + \frac{\mathbf{q}}{m}\right)\cdot \mathbf{E}\mathrm{Re}D(\mathbf{q}, \omega)\frac{\partial S_0(\epsilon + \omega)}{\partial \epsilon}\mathrm{Im}G_0^A(\mathbf{p} + \mathbf{q}, \epsilon + \omega).\end{aligned} \tag{17}$$

The second term in Eq. (8) corresponds to the part of ϕ_1 which originates from the nonlocal part of collision integral [the Poisson bracket term in Eq. (12)],

$$\begin{aligned}\phi_1(\mathbf{p}, \epsilon) &= \frac{\tau}{2}\{\Sigma^A_{e\text{-ph}} + \Sigma^R_{e\text{-ph}}, S_0\}\\ &= \frac{\partial S_0(\epsilon)}{\partial \epsilon}\int \frac{d^4Q}{(2\pi)^4}|g_q|^2\tau e\left(\mathbf{v} + \frac{\mathbf{q}}{m}\right)\cdot \mathbf{E}\mathrm{Re}D(\mathbf{q}, \omega)S_0(\epsilon + \omega)\,\mathrm{Im}[G_0^A(\mathbf{p} + \mathbf{q}, \epsilon + \omega)]^2.\end{aligned} \tag{18}$$

Performing calculations in Eq. (16) we obtain the result reported in Eq. (1).

Experimental situation.—Taking into account the result of the present paper, the resistivity of an impure metal can be written as

$$\rho = \rho_0 + \rho_{\mathrm{ren}} + \rho_{e\text{-ph}} + \rho_{e\text{-}e} + \rho_{e\text{-ph}-\mathrm{imp}}, \tag{19}$$

where $\rho_0 = 1/\sigma_0$ is the Drude resistivity, the bare residual resistivity due to electron-impurity scattering; $\rho_{\mathrm{ren}} = -(\sigma - \sigma_0)/(\sigma_0)^2$ is the phonon renormalization of the Drude resistivity, calculated in this paper; and $\rho_{e\text{-ph}}$ is the Bloch-Gruneisen term. The last two terms describe the interference between interactions in an impure metal: $\rho_{e\text{-}e}$ is the contributions of the weak localization and of the interference between electron-electron and electron-impurity scattering [10] which are important at helium temperatures. For temperatures somewhat higher the interference between electron-phonon and electron-impurity scattering dominates [11],

$$\frac{\rho_{e\text{-ph}-\mathrm{imp}}}{\rho_0} = \left[2\left(\frac{u_l}{u_t}\right)^3 - 1 + \frac{\pi^2}{16}\right]\frac{4\beta T^2}{\epsilon_F p_F u_l}\int_0^{\theta_D/T}dx\, x\left(\frac{1}{\exp(x) - 1} - \frac{x\exp(x)}{[\exp(x) - 1]^2}\right). \tag{20}$$

If $T \leq \omega_D/5$, the integral in Eq. (20) tends to $-\pi^2/6$. In Eq. (20), β is the kinetic constant of the electron-phonon interaction [12],

$$\beta = \left(\frac{2}{3}\epsilon_F\right)^2\frac{\nu(0)}{2\mu u_l^2} = \lambda\left(\frac{2p_F}{q_D}\right)^2, \tag{21}$$

where $\nu(0) = mp_F/\pi^2$ is the electron both-spin density of states and μ is the mass density of the material.

The Bloch-Gruneisen correction (see Ref. [7]) is

$$\frac{\rho_{e\text{-ph}}}{\rho_0} = \frac{\pi}{2}\frac{\beta\tau T^5}{(p_F u_l)^4}\int_0^{\theta_D/T}dx\frac{x^5}{[\exp(x) - 1][1 - \exp(-x)]}. \tag{22}$$

Under the basic assumption of this work, that the dominating mechanism of the electron momentum relaxation is electron-impurity scattering ($\rho_{e\text{-ph}} \ll \rho_0$), Eq. (22) was obtained in Ref. [13] by the quantum transport equation.

Obviously, in this case Eq. (22) corresponds to a "weak" Mattheissen's rule. Note also that Eq. (22) may be obtained solely from terms proportional to $\mathrm{Im}D^R(\mathbf{q},\omega)$ of the third diagram in Fig. 1. These terms correspond to the quasiparticle approximation in the transport equation [14], while terms with $\mathrm{Re}D^R(\mathbf{q},\omega)$, which result in the renormalization, originate from quantum corrections to the transport equation.

The experimental temperature dependencies of the resistivity of an impure metal [15] are well described by the interference and the Bloch-Gruneisen terms [Eqs. (20) and (22)] in the wide temperature interval 3–300 K. In materials with relatively strong electron-impurity scattering ($\tau \sim 3$ fs) the interference contribution dominates over the Bloch-Gruneisen term up to 150 K (see Ref. [16]), but the renormalization of resistivity is small due to inequality $(\omega_D)^{-1} \sim (300\ \mathrm{K})^{-1} \sim 30\ \mathrm{fs} > \tau$. In relatively pure materials with $\tau \sim 100$ fs, where the renormalization of the resistivity is significant, the Bloch-Gruneisen term prevails already for temperatures from 10 K (see Ref. [17]). In the interval from helium to the Debye temperature the change of the renormalization term is of the order of $\lambda\rho_0$, while the change of the Bloch-Gruneisen term is $\sim(\omega_D\tau)\rho_0 \gg \rho_0$. Therefore, it is difficult to extract the renormalization effect from the resistivity temperature dependence. One possible way to study the renormalization effect would be to determine the bare residual resistivity from measurements of the T^2 term using Eq. (20). Comparison with the experimentally determined renormalized residual resistivity would give the renormalization factor. Alternatively, the conductivity renormalization might be investigated via its strong dependence on the electron-impurity scattering rate. As we have already discussed, for pure materials ($\omega_D\tau > 1$) the renormalization is significant, whereas when the parameter $\omega_D\tau$ decreases the renormalization falls rapidly to zero.

In conclusion, this paper demonstrates that the Drude conductivity is significantly renormalized by the electron-phonon interaction. We calculated the renormalization using both the linear response and quantum transport equation approaches. In the transport equation method the renormalization arises through corrections to the nonequilibrium electron density of states [Eq. (17)] and the nonlocal part of the electron-phonon collision integral [Eq. (18)]. Reference [1] did not consider such terms in the renormalization of the Bloch-Gruneisen conductivity. In the light of this work we expect the renormalization of other kinetic coefficients.

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