

## Molecular-field approximation for Josephson-coupled superconducting arrays in a magnetic field

Wan Y. Shih and D. Stroud

Department of Physics, The Ohio State University, Columbus, Ohio 43210

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A molecular-field approximation is developed for two-dimensional or three-dimensional Josephson- or proximity-coupled superconducting arrays in an applied magnetic field. The molecular-field transition temperature  $T_c(H)$  is found to map onto the highest energy of an electron in a tight-binding band in the presence of a magnetic field (a problem treated by Hofstadter).  $T_c(H)$  is thus periodic in  $H$  with complex substructure, some of which has been observed experimentally.

Two-dimensional arrays of weakly coupled superconducting grains have recently been found to show remarkable behavior in an applied perpendicular magnetic field.<sup>1-3</sup> Such arrays can be made of small ( $\sim 10 \mu\text{m}$  diameter) lead disks, for example, arranged on a lattice and embedded in a normal metal host, which then acts to couple the disks via the proximity effect, or from a periodic array of superconducting "crosses" with superconductor-insulator-superconductor ( $S-I-S$ ) junctions on a periodic lattice. In brief, many measurable properties of such arrays [resistivity above the array transition temperature  $T_c(H)$ , low-temperature critical current] seem to be periodic functions of magnetic field, with a period of one flux quantum per unit cell of the grain lattice and a great deal of fine structure.

The purpose of this Rapid Communication is to describe a simple molecular-field treatment of the phase transition in such a two-dimensional array, which accounts for the gross features of the observed behavior in a qualitative way. While a molecular-field approximation is certainly not adequate for understanding phase transitions in two-dimensional systems, nonetheless it is useful to have such a model available as a standard for comparison with more exact treatments, for extracting simple estimates of various measurable quantities, and for use in three-dimensional systems. Our results thus supplement the recent elegant Monte Carlo calculations of Teitel and Jayaprakash,<sup>4</sup> who demonstrate the existence of a phase transition in these arrays at certain special values of applied magnetic fields.

The basis of our calculations is the following model for the interaction energy of an array of  $N$  weakly coupled superconducting grains:

$$\mathcal{X} = - \sum_{\langle ij \rangle} J_{ij} \cos(\phi_i - \phi_j - A_{ij}) . \quad (1)$$

Here  $J_{ij}$  is the coupling energy between the  $i$ th and  $j$ th grains, given in terms of the critical current by  $J_{ij} = (\hbar/2e)I_{ij}$ ;  $\phi_i$  is the phase of the order parameter on the  $i$ th grain; and  $A_{ij}$  is given in terms of the vector potential  $A(x)$  by

$$A_{ij} = \frac{2e}{\hbar c} \int_{\vec{x}_i}^{\vec{x}_j} \vec{A} \cdot d\vec{l} , \quad (2)$$

where  $\vec{x}_i$  is the position of the center of the  $i$ th grain. The sum in (1) runs over all pairs of Josephson- or proximity-coupled grains. In writing (1) we have neglected charging effects<sup>5</sup> (that is, we assume infinite grain capacitance). We

also assume that the phase is constant across each grain ("point grains") so that the integral in (2) is well defined. Wide junctions and finite grain sizes will render the model (1) somewhat inaccurate in practice. The final approximation we make is to ignore the temperature dependence of  $J_{ij}$ , i.e., of the critical currents  $I_{ij}$ . In reality, this temperature dependence is known and can easily be incorporated.<sup>6</sup> Our neglect should not affect the results if  $T_c(H)$  is well separated from the transition temperature  $T_{c0}$  of the individual grains.

In principle, of course,  $A_{ij}$  should refer to the *local* magnetic field, including that produced by Josephson screening currents, and not just the applied field. But these induced fields are difficult to incorporate in practice, as they depend very much on sample geometry and boundary effects. The calculations reported here therefore neglect these screening fields. This neglect is probably justified in the "weak-coupling limit" in which the  $J_{ij}$ 's are small and the Josephson penetration depth is large compared with intergrain spacing.

The thermodynamic properties of (1) are obtained by treating the phases of  $\phi_i$ 's as classical thermodynamic variables within the canonical ensemble. Thus the expectation value of an operator  $O$  is

$$\langle O \rangle = \frac{1}{Z} \int \left[ \prod_{i=1}^N d\phi_i \right] O(\phi_1, \dots, \phi_N) e^{-\beta \mathcal{X}} , \quad (3)$$

where the partition function  $Z$  is

$$Z = \int \left[ \prod_{i=1}^N d\phi_i \right] e^{-\beta \mathcal{X}} , \quad (4)$$

where  $\beta = (k_B T)^{-1}$  is the inverse temperature and  $N$  is the number of grains. The molecular-field approximation is then expressed by the condition

$$\eta_i = \langle e^{i\phi_i} \rangle \sim Z_i^{-1} \int_0^{2\pi} d\phi_i e^{i\phi_i} e^{-\beta \mathcal{X}_{\text{eff}}^{(i)}} , \quad (5)$$

$$Z_i = \int_0^{2\pi} d\phi_i e^{-\beta \mathcal{X}_{\text{eff}}^{(i)}} , \quad (6)$$

$$\mathcal{X}_{\text{eff}}^{(i)} = - \sum_{j \neq i} J_{ij} [ \cos\phi_i \langle \cos(\phi_j + A_{ij}) \rangle + \sin\phi_i \langle \sin(\phi_j + A_{ij}) \rangle ] , \quad (7)$$

where the expectation values in (7) are to be calculated self-consistently from equations of the form (5). Thus Eqs.

(5)–(7) constitute a set of  $N$ -coupled nonlinear complex equations (or  $2N$  real equations) in the  $N$  unknown “phase-order parameters”  $\eta_i$ , and, in general, must be solved numerically. Note that as written the equations are applicable to both ordered and disordered arrays and to three-dimensional systems as well as two-dimensional ones.

We have solved Eqs. (5)–(7) numerically for two-dimensional square lattices of coupled superconducting grains in the presence of various applied fields, using the gauge  $A = Hx\hat{y}$ . With this choice, all the bonds in the  $x$  direction are field independent, while those in the  $y$  direction include a phase factor  $A_{ij} = 2\pi n f \text{sgn}(x_j - x_i)$ , where  $f$  is the value of the flux per unit cell in units of an elementary flux quantum  $\phi_0 = hc/2e$ , and  $n$  is the  $x$  coordinate of the bond, measured in units of the intergranular spacing  $a$ . For rational values of  $f$ , say  $f = p/q$ , we assumed the ordered state to be periodic on an  $(mq) \times (mq)$  unit cell, with  $m \geq 1$  an integer, and solved the resulting  $(mq)^2$  coupled equations by iteration. The right-hand side of (7) can be expressed in terms of standard modified Bessel functions of the first kind, and this facilitates the calculation. Various different starting configurations for the iteration were tried, and the resulting transition temperatures were always the same. For  $q \geq 3$  several final configurations were found which could not be transformed into each other simply by an overall rotation of the phase or a translation, and for  $q \geq 4$  these sometimes corresponded to different ground-state energies (i.e., to metastable configurations). The states obtained by iteration were always found to be periodic with a  $q \times q$  unit cell. The metastable configurations found by our method are presumably obtained because of the similarity of our problem to a spin-glass, in which similar metastable configurations have been found by mean-field theory.<sup>7</sup>

Figures 1 and 2 show the mean-field transition temperature  $T_c(f)$  and mean-field phase configurations for a number of values of  $f$ . (The results are symmetric about  $f = \frac{1}{2}$ ). The lines connecting the calculated points in Fig. 1 are merely to guide the eye; in actuality, the real curve obtained from Eqs. (5)–(7) will show a high degree of variation on a small scale, as discussed below. Note, in particular, the maximum in  $T_c(f)$  at  $f = \frac{1}{2}$ , a feature that has been seen experimentally by several groups.<sup>2,3</sup> Quantitatively, however, the dip in  $T_c$  is not as large as that seen in Monte Carlo simulations<sup>4</sup> by Teitel and Jayaprakash. The mean field  $T_c$  at  $f = \frac{1}{3}$  falls below that at  $f = \frac{1}{2}$ , in agreement with the Monte Carlo results.<sup>6</sup> The value of  $T_c(f)$  for  $f = 1/q$  and  $q$  large approaches  $T_c(0)$ , however, in disagreement with the result of Teitel and Jayaprakash that  $T_c(1/q)$  is bounded above by  $C/q$  where  $C$  is a field-independent constant. This suggests that the mean-field theory may represent a transition into an ordered state that is approached at  $T_c^{\text{MF}}(f)$  but in reality prevented from actually stabilizing by two-dimensional fluctuations that are not properly included in the mean-field theory.

The mean-field equation for  $T_c(f)$  can be mapped onto a totally unrelated problem by a simple transformation. Near  $T_c$ , the various  $\eta_i$ 's in Eq. (5) are all small and so, therefore, are the expectation values appearing in (7). We may therefore expand the exponential in (5) as  $1 - \beta H_{\text{eff}}$ , keeping the first term since the zeroth-order integral vanishes. Carrying out the phase integral then leads to the *linearized*

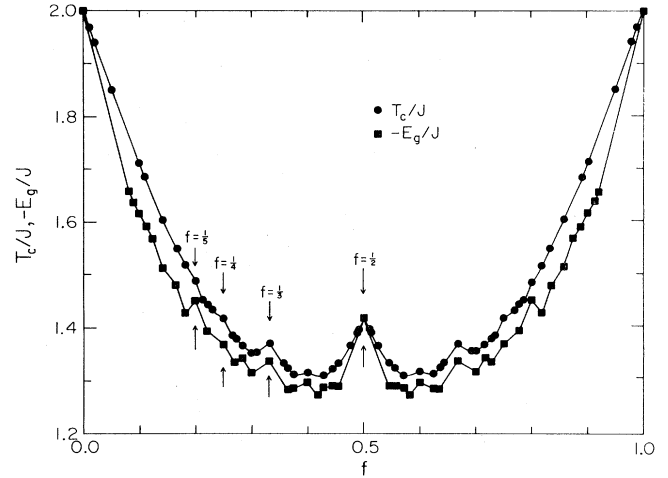


FIG. 1. Molecular-field transition temperature  $T_c^{\text{MF}}(f)$  and negative ground-state energy  $-E_g(f)$  as a function of  $f$ , the magnetic field measured in units of elementary flux quanta per unit cell, for a square lattice of Josephson—or proximity—coupled superconducting grains.  $T_c^{\text{MF}}(f)$  and  $-E_g(f)$  for a three-dimensional simple cubic lattice are obtained from the two-dimensional result simply by adding  $J$  at each value of  $f$ .

mean-field equation

$$\eta_i - \frac{\beta}{2} \sum_j J_{ij} e^{iA_{ij}} \eta_j = 0 \quad (8)$$

This is precisely the Schrödinger equation in the tight-binding representation for an electron (of charge  $2e$ ) in a magnetic field  $H$  and moving in a lattice of  $N$  sites. The or-

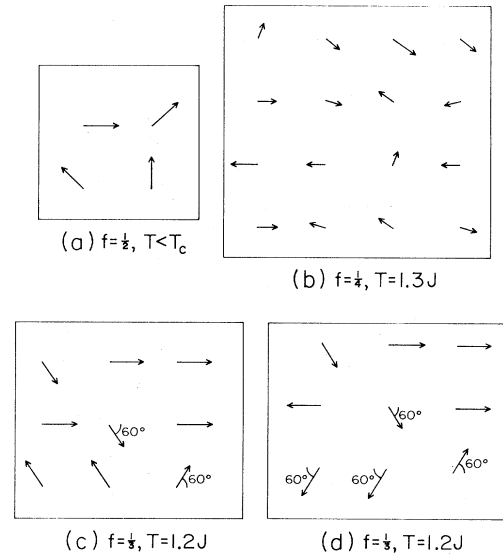


FIG. 2. Phase-order parameter configurations near  $T_c^{\text{MF}}(f)$  for  $f = \frac{1}{2}$ ,  $\frac{1}{3}$ , and  $\frac{1}{4}$ . The lengths and directions of the arrows represent the complex numbers  $\eta_i$ . For  $f = \frac{1}{3}$ , two degenerate configurations are shown.

der parameter  $\eta_i$  represents the complex wave function at the  $i$ th site,  $J_{ij}$  is the hopping integral, and  $\beta^{-1} = k_B T$  is the energy eigenvalue.  $T_c$  is the highest value of  $T$  for which Eq. (8) has a nontrivial solution.

For the two-dimensional lattice we consider, the corresponding tight-binding problem has been worked out in detail by Hofstadter.<sup>8</sup> Our curve for  $T_c^{\text{MF}}(f)$  should correspond, suitably scaled, to the variation of his band edge with magnetic field. We have checked that the two do indeed agree. Note also that our phase configurations near  $T_c$  correspond to the wave function of this band-edge states.

Also shown in Fig. 1 is the ground-state energy  $-E_g(f)$  as calculated from Eqs. (5)–(7). At  $T=0$  these equations lead to  $\eta_i$ 's which all have unit magnitude and vary only in direction; once these are found,  $E_g(f)$  is obtained from the equation

$$E_g(f) = - \sum_{\langle ij \rangle} J_{ij} \text{Re}(\eta_i^* \eta_j e^{iA_{ij}}) . \quad (9)$$

The value of  $E_g(f)$  obtained from these mean-field equations is *exact*, as is clear from the derivation and as we have checked by comparing our results with the Monte Carlo simulations. Note the highly nonmonotonic behavior of  $E_g(f)$  vs  $f$ ; the general features of the curve tend to reflect those of  $T_c^{\text{MF}}(f)$  but do not match them exactly.  $-E_g(f)$ , in general, lies slightly below  $T_c^{\text{MF}}(f)$ , probably because  $E_g(f)$  is calculated with the additional constraint that  $\eta_i$ 's have unit magnitude.

We conclude with some qualitative remarks. The molecular-field approximation should be more accurate in three dimensions than in two, and ought to be particularly useful for treating disordered granular systems in a magnetic field. Presumably, oscillations in such properties as critical current which have been reported in the literature<sup>9</sup> are due

to periodic variations in Josephson coupling with magnetic field, and the diminution in these oscillations with increasing magnetic field is caused by variations in the areas of superconducting loops, which would make the various periods of oscillation fluctuate randomly from one loop to another. Note also that for *ordered* simple cubic arrays, with  $\vec{H}$  parallel to a crystal axis,  $T_c^{\text{MF}}(f)$  is obtained in the molecular-field approximation from the two-dimensional result (Fig. 1) simply by *adding*  $J$  at each  $f$ , and so is  $-E_g(f)$ . This simple behavior is predicted because the *ordered* configuration predicted by Eqs. (5)–(7) turns out to be identical along each line of grains parallel to  $\vec{H}$ , and given by the two-dimensional configurations such as those shown in Fig. 2. Note that, in three-dimensional samples, one expects both a longitudinal and a transverse magnetoresistance above  $T_c(f)$ , both varying periodically with  $f$ . Finally, we remark that in a *triangular* lattice, such as that investigated by Resnick *et al.*,<sup>1</sup>  $T_c^{\text{MF}}(f)$  will show a much less conspicuous maximum at  $f = \frac{1}{2}$  than does the square lattice, as is clear from the calculations of Claro and Wannier<sup>10</sup> for the analogous tight-binding electronic problem.

We hope to return to these and other questions in a future publication.

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