Physics 880.06: Problem Set 3

Note: please turn these problems into the mailbox of the grader, Wissam Al-Saidi, by 5 P. M. on Friday, April 25. Each problem is worth 10 points.

1. The "Bogoliubon" operators were introduced in class by the transformation

$$c_{\mathbf{k}\uparrow} = u_{\mathbf{k}}\gamma_{\mathbf{k},0} + v_{\mathbf{k}}\gamma^{\dagger}_{\mathbf{k},1} \tag{1}$$

$$c^{\dagger}_{-\mathbf{k}\downarrow} = -v^{*}_{\mathbf{k}}\gamma_{\mathbf{k},0} + u_{\mathbf{k}}\gamma^{\dagger}_{\mathbf{k},1}, \tag{2}$$

where the coefficients $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ satisfy the normalization condition

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1. \tag{3}$$

Show that the operators $\gamma_{,\mathbf{k},0}$, $\gamma_{\mathbf{k},1}$ and their Hermitean conjugates satisfy the standard Fermi anticommutation relations

$$[\gamma^{\dagger}_{\mathbf{k},i}, \gamma_{\mathbf{k},j}]_{+} = \delta_{ij}, \tag{4}$$

where i and j can take on the values 1 and 2, and $[..,...]_{+}$ denotes an anticommutator.

2. The operator $a^{\dagger}_{\mathbf{k}} = c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}$ creates a pair of electrons with opposite wave vector and opposite spin.

   Show that if $\mathbf{k} \neq \mathbf{k}'$, the commutator (not the anticommutator) $[a^{\dagger}_{\mathbf{k}}, a_{\mathbf{k}'}] = 0$, as expected for Bose operators, but that if $\mathbf{k} = \mathbf{k}'$, this commutator does not equal unity, as one would expect for Bose operators, and calculate $[a^{\dagger}_{\mathbf{k}}, a_{\mathbf{k}}]$.

3. The electronic specific heat of the Bogoliubons is, as noted by Tinkham,

$$C_{es} = T\frac{dS_{es}}{dT}, \tag{5}$$

where S_{es} is the entropy of the Bogoliubons and is given by the standard result for a collection of Fermions:

$$S = -2k_B \sum_{\mathbf{k}}[(1 - f_{\mathbf{k}})ln(1 - f_{\mathbf{k}}) + f_{\mathbf{k}}lnf_{\mathbf{k}}]. \tag{6}$$

Here

$$f_{\mathbf{k}} = \frac{1}{e^{\beta E_{\mathbf{k}}} + 1} \tag{7}$$

and

$$E_{\mathbf{k}} = \sqrt{\Delta^2 + \xi_{\mathbf{k}}^2}, \tag{8}$$

Δ being the energy gap, which we assume is real, and $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - E_F$.

Show that the specific heat can be written as

$$C_{es} = 2\beta k_B \sum_{\mathbf{k}} \left(-\frac{\partial f_{\mathbf{k}}}{\partial E_{\mathbf{k}}} \right) \left(E_{\mathbf{k}}^2 + \frac{1}{2}\beta \frac{d\Delta^2}{d\beta} \right), \tag{9}$$

where $\beta = 1/(k_B T)$.

Show explicitly that, if Δ is temperature-independent, the specific heat varies at low temperatures as $\exp(-\Delta/k_B T)$ multiplied by a function which varies more slowly with temperature. (This is also true even if Δ is dependent on temperature, provided that it remains finite at T = 0.)