

Physics 880.06: Problem Set 2

Note: please turn these problems into the mailbox of the grader, Wissam Al-Saidi, by 12 noon on Friday, April 18.

- Consider a free electron gas of density n electrons per unit volume (in three dimensions). Calculate the electronic density of states $N(E)$ (in states per unit energy per unit volume). Show, in particular that $N(E) = A\sqrt{E}$, and calculate the constant A in terms of fundamental constants (i. e. \hbar , electron mass, and electronic charge, if they enter) and the density n .
 - Calculate the Fermi energy E_F in terms of n .
 - Hence, find $N(E_F)$.
 - Find the actual numerical values of these quantities for the metal aluminum, assuming that there are three valence electrons per atom.
 - Repeat (a) - (c) for a two-dimensional electron gas, except that n is understood now as the electron density per unit area.
- We showed in class that the relative wave function of a Cooper pair is proportional to

$$\sum_{k>k_F} \frac{\cos \mathbf{k} \cdot \mathbf{r}}{2\xi_{\mathbf{k}} + 2E_F - E}, \quad (1)$$

where (as stated in class) $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - E_F$, $E = 2E_F - 2\hbar\omega_c \exp[-1/(N(0)V)]$, $\hbar\omega_c$ is a suitable cutoff energy, E_F is the Fermi energy, and the sum runs up to \mathbf{k} such that $\epsilon_{\mathbf{k}} = E_F + \hbar\omega_c$. If we define $g_{\mathbf{k}} = 1/[2\xi_{\mathbf{k}} + 2E_F - E]$, then the largest coefficient $g_{\mathbf{k}}$ corresponds to $k = k_F$.

- Estimate the value of $\epsilon_{\mathbf{k}}$ such that $g_{\mathbf{k}}$ equals half its maximum value, and estimate the corresponding value of k . Assume that $\hbar\omega_c$ is much less than E_F . Show that in the weak-coupling regime, this energy satisfies $\epsilon_{\mathbf{k}} = E_F + \Delta$, where Δ is a quantity much less than $\hbar\omega_c$.
- Hence, obtain an approximate expression for the *coherence length* (i. e. the size of the Cooper pair in space). Make use of an appropriate uncertainty relation between this size and the spread in k -space of the

wave vectors entering the Cooper pair wave function. Make a numerical estimate of this coherence length for aluminum, taking Δ comparable to $k_B T_c$, where T_c is the superconducting transition temperature for aluminum and using a suitable value of the Fermi energy.

3. (20 pts.) Using the BCS ground state wave function described in class, confirm the statement of Tinkham (and made by me in class) that the mean-square fluctuation in electron number in this wave function is

$$\langle (N - \bar{N})^2 \rangle = 4 \sum_{\mathbf{k}} u_{\mathbf{k}}^2 v_{\mathbf{k}}^2. \quad (2)$$

where \bar{N} is the expectation value of the number of electrons in this wave function, given by $\bar{N} = 2 \sum_{\mathbf{k}} v_{\mathbf{k}}^2$.

You may take the coefficients $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ to be real.

Hint: first show that

$$\langle \Psi_G | n_{\mathbf{k}\sigma} n_{\mathbf{k}'\sigma'} | \Psi_G \rangle = \langle \Psi_G | n_{\mathbf{k}\sigma} | \Psi_G \rangle \langle \Psi_G | n_{\mathbf{k}'\sigma'} | \Psi_G \rangle \quad (3)$$

for $\mathbf{k} \neq \mathbf{k}'$, and, for $\mathbf{k} = \mathbf{k}'$, it equals

$$\langle \Psi_G | n_{\mathbf{k}\sigma} | \Psi_G \rangle. \quad (4)$$

(Here, σ is a spin index and $|\Psi_G\rangle$ denotes the ground state. With these results, you should be able to show that $\langle (N - \bar{N})^2 \rangle$ simplifies to a single sum.