Physics 880.06: Problem Set 7

Note: please turn these problems into the mailbox of the grader, Wissam Al-Saidi, by the beginning of class on Thursday, June 6.

1. (20 pts.) As stated in class, the *anisotropic Ginzburg-Landau free energy density* is given by the following expression:

$$f = f_{n0} + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m_{ab}}\left|\left(-i\hbar\nabla - \frac{e^*\mathbf{A}}{c}\right)_\perp \psi\right|^2 + \\ + \frac{1}{2m_c}\left|\left(-i\hbar\nabla_z - \frac{e^*A_z}{c}\right)\psi\right|^2 + \frac{|\mathbf{B}|^2}{8\pi}. \tag{1}$$

Here α and β are scalar coefficients, ψ is the complex scalar order parameter, f is the free energy per unit volume, f_{n0} is the normal state free energy density, $\mathbf{A}$ is the vector potential, $\mathbf{B}$ is the magnetic induction, and m_{ab} and m_c are the effective masses. We also assume that the coefficient α has the temperature dependence

$$\alpha = \alpha'[T/T_c(0) - 1], \tag{2}$$

where α' is a positive constant and $T_c(0)$ is the superconducting transition temperature at zero field.

The two Ginzburg-Landau equations obtained from this free energy are

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m_{ab}}\left(-i\hbar\nabla - \frac{e^*\mathbf{A}}{c}\right)_\perp^2 \psi + \\ + \frac{1}{2m_c}\left(-i\hbar\nabla_z - \frac{e^*A_z}{c}\right)^2 \psi = 0 \tag{3}$$

and

$$\mathbf{J} = \frac{e^*}{2m_{ab}}\left[\psi^*\left(-i\hbar\nabla - \frac{e^*\mathbf{A}}{c}\right)_\perp \psi + c.c.\right] + \\ + \frac{e^*}{2m_c}\left[\psi^*\left(-i\hbar\nabla_z - \frac{e^*A_z}{c}\right)\psi + c.c.\right]\hat{z}. \tag{4}$$

(a). (10 pts.) By linearizing Eq. **??** with respect to ψ, and finding the energy of the lowest Landau level, find the transition temperature

$T_c(B)$ for (i) $\mathbf{B}$ parallel to the z axis; (ii) $\mathbf{B}$ parallel to the x axis. Assume that the field $\mathbf{B}$ is specified. Your answers should involve the effective masses m_{ab} and m_c.

(b). (10 pts.) Assuming that the order parameter ψ is independent of position, obtain an expression for the current density $\mathbf{J}$ in terms of the vector potential $\mathbf{A}$. From this equation, and Ampere's Law, find a differential equation which describes the variation of the magnetic field with position, assuming that $\mathbf{B} = B(x)\hat{z}$. What is the penetration depth corresponding to this field variation?

Note: in the above problem, I am using the directions x, y, and z interchangeably with a, b, and c.

2. In this problem, you will make a very rough estimate of the rate of quantum tunneling of the phase out of a small Josephson junction, using the WKB approximation. As stated in class, if one has a particle of mass m and energy E in a one-dimensional well V(x), then the probability per unit time that the particle will escape from the well is approximately

$$w = \frac{\omega_0}{2\pi} \exp\left[-\frac{2}{\hbar}\int_{x_1}^{x_2} p dx\right]. \tag{5}$$

Here, $p = \sqrt{2m[V(x) - E]}$ is the particle momentum, and x_1 and x_2 are the lower and upper limits of the classically forbidden region for a particle of energy E. Finally, ω_0 is the oscillation frequency for a particle in the well, so that $\omega_0/(2\pi)$ is the frequency at which the particle "attempts" to escape from the well.

(a). For a phase "particle" in a Josephson junction, the Hamiltonian is

$$H = -E_J \cos\phi - E_C \frac{\partial^2}{\partial \phi^2}, \tag{6}$$

where $E_C = (2e)^2/(2C)$ and C is the junction capacitance. Write out eq. (**??**) explicitly for this case. What is the frequency ω_c if the particle is near the bottom of the well?

(b). Suppose that the energy is very close to the potential energy minimum ($E \sim -E_J$). What is the width of the classically forbidden region in this case (assuming tunneling in the positive ϕ direction)? Calculate w in this case, in terms of E_J and E_C.

3. In Problem Set 4, you considered the properties of a two-dimensional superconductor in which the excitation spectrum satisfies

$$E(\mathbf{k} = \sqrt{C_1(\delta k)_1^2 + C_2(\delta k)_2^2}, \tag{7}$$

where δk is a two-component vector, with components $(\delta k)_1$ and $(\delta k)_2$, describing the distance of the $\mathbf{k}$ vector from the node point.

In this problem, you will consider this same excitation spectrum in the presence of a magnetic field $\mathbf{B}$ perpendicular to the plane.

To calculation the spectrum, make the substitution $\delta\mathbf{k} \to -i\hbar\nabla$, where ∇ is the two-dimensional gradient operator. In the presence of a vector potential $\underline{A}$, the correct substitution is $\delta\mathbf{k} \to -i\hbar\nabla - \frac{e^*\mathbf{A}}{c}$.

(a). What is the Hamiltonian in the presence of a $\mathbf{B}$ field $\mathbf{B} = B\hat{z}$?

(b). Show that the eigenvalues of this spectrum are given by $\mathbf{E}_n = K\sqrt{(n+\frac{1}{2})B}$, $n = 0, 1, ...,$ and find the constant K in terms of C_1 and C_2. [Thus, there is a gap in the spectrum proportional to $\sqrt{B}$.]

4. In one model for Cooper pairs in the high-T_c superconductors, the center-of-mass wave function for the pairs (in a two-dimensional CuO_2 layer) is said to be a $d_{x^2-y^2}$ state. That is, it is made of a certain linear combination of Cu 3d states which has an angular dependence of the form $x^2 - y^2)/r^2$. The full center-of-mass wave function thus has the form $f(r)(x^2 - y^2/r^2)$, where f(r) depends only on the radial coordinate r.

Look up the spherical harmonics Y_{2m} which describe the angular dependence of a d-state, and show that there is a specific linear combination of these which is proportional to $(x^2 - y^2)/r^2$.

5. In class I gave a way of estimating the energy to create a vortex in a two-dimensional Josephson junction array of linear dimension L. In this problem you will do a similar calculation to estimate this energy for a thin, homogeneous superconducting film. Let the film have thickness d, and let it have a superconducting order parameter ψ. The Ginzburg-Landau free energy per unit volume is (ignoring the field energy)

$$f - f_{n0} = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}| - i\hbar\nabla\psi|^2. \tag{8}$$

Here α, β, and m are the usual Ginzburg-Landau free energy parameters, and f_{n0} is the free energy density of the reference normal state.

Assume that the vortex state is characterized by an order parameter

$$\psi = \psi_0 e^{i\theta} \tag{9}$$

where $\psi_0 = \sqrt{-\alpha/\beta}$ and θ is the polar angle. Thus, the phase of the superconducting order parameter satisfies $\phi = \theta$.

Calculate the extra energy associated with the presence of a vortex. Do the calculation by integrating the extra free energy from some minimum distance a from the vortex to some maximum distance L (comparable to the linear dimension of the film). Note that in this approximation only the gradient term contributes to the extra free energy. You should get a result of the form

$$E_v = C\ln(L/a), \tag{10}$$

and find C. Express C in terms of α, β, m, and the film thickness d, and hence in terms of the penetration depth λ, coherence length xi, and thermodynamic critical field H_c.

6. *This problem is for edification only: not to be turned in.*

In class we discussed a SQUID consisting of two Josephson junctions, with critical currents I_{c1} and I_{c2}. We showed, for the case $I_{c1} = I_{c2}$, that the critical current of the SQUID was a periodic function of the flux Φ through the loop with a period Φ_0, where $\Phi_0 = hc/2e$ is the flux quantum, and maxima at $\Phi = n\Phi_0$.

Now in high-T_c superconductors with $d_{x^2-y^2}$ order parameters, it is possible to have *two* types of junctions. One is a "zero" junction, which has current-phase relation $I = I_c \sin\gamma$, where γ is the gauge-invariant phase difference across the junction. The other is a "pi" junction, in which $I = I_c \sin(\gamma - \pi) = -I_c \sin\gamma$, where I_c is a positive number. Whether a particular junction is a zero or a pi junction depends on exactly how the two crystals which make up the junction are cut (i. e. what crystallographic face is parallel to the surface).

Show that if one has a SQUID made out of two junctions, one a zero junction and one a pi junction, characterized by the same I_c, then the critical current of the loop is periodic in the flux through the junction with period Φ_0, and maxima at $(n + 1/2)\Phi_0$ where n is a positive or negative integer.