

Physics 880.06: Problem Set 2

Note: please turn these problems into the mailbox of the grader, Wissam Al-Saidi, by 12 noon on Friday, April 19.

- Consider a free electron gas of density n electrons per unit volume (in three dimensions). Calculate the electronic density of states $N(E)$ (in states per unit energy per unit volume). Show, in particular that $N(E) = A\sqrt{E}$, and calculate the constant A in terms of fundamental constants and the density n .
 - Calculate the Fermi energy E_F in terms of n .
 - Hence, find $N(E_F)$.
 - Find the actual numerical values of these quantities for the metal aluminum, assuming that there are three valence electrons per atom.
 - Repeat (a) - (c) for a two-dimensional electron gas, except that n is understood now as the electron density per unit area.
- We showed in class that the relative wave function of a Cooper pair is proportional to

$$\sum_{k>k_F} \frac{\cos \mathbf{k} \cdot \mathbf{r}}{2\xi_{\mathbf{k}} + 2E_F - E}, \quad (1)$$

where (as stated in class) $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - E_F$, $E = 2E_F - 2\hbar\omega_c \exp[-1/(N(0)V)]$, $\hbar\omega_c$ is a suitable cutoff energy, E_F is the Fermi energy, and the sum runs up to \mathbf{k} such that $\epsilon_{\mathbf{k}} = E_F + \hbar\omega_c$. If we define $g_{\mathbf{k}} = 1/[2\xi_{\mathbf{k}} + 2E_F - E]$, then the largest coefficient $g_{\mathbf{k}}$ corresponds to $k = k_F$.

- Estimate the value of $\epsilon_{\mathbf{k}}$ such that $g_{\mathbf{k}}$ equals half its maximum value, and estimate the corresponding value of k . Assume that $\hbar\omega_c$ is much less than E_F . Show that in the weak-coupling regime, this energy satisfies $\epsilon_{\mathbf{k}} = E_F + \Delta$, where Δ is a quantity much less than $\hbar\omega_c$.
- Hence, obtain an approximate expression for the *coherence length* (i. e. the size of the Cooper pair in space). Make use of an appropriate uncertainty relation between this size and the spread in k -space of the wave vectors entering the Cooper pair wave function. Make a numerical estimate of this coherence length for aluminum, taking Δ comparable

to $k_B T_c$, where T_c is the superconducting transition temperature for aluminum and using a suitable value of the Fermi energy.

3. . (20 pts.) Using the BCS ground state wave function described in class, confirm the statement of Tinkham that the mean-square fluctuation in electron number in this wave function is

$$\langle (N - \bar{N})^2 \rangle = 4 \sum_{\mathbf{k}} u_{\mathbf{k}}^2 v_{\mathbf{k}}^2. \quad (2)$$

where \bar{N} is the mean number of electrons in this wave function, given by $\bar{N} = 2 \sum_{\mathbf{k}} v_{\mathbf{k}}^2$.

You may take the coefficients $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ to be real.

Hint: first show that

$$\langle \Psi_G | n_{\mathbf{k}\sigma} n_{\mathbf{k}'\sigma'} | \Psi_G \rangle = \langle \Psi_G | n_{\mathbf{k}\sigma} | \Psi_G \rangle \langle \Psi_G | n_{\mathbf{k}'\sigma'} | \Psi_G \rangle \quad (3)$$

for $\mathbf{k} \neq \mathbf{k}'$, and, for $\mathbf{k} = \mathbf{k}'$, it equals $\langle \Psi_G | n_{\mathbf{k}\sigma} | \Psi_G \rangle$. (Here, σ is a spin index and $|\Psi_G\rangle$ denotes the ground state. With these results, you should be able to show that $\langle (N - \bar{N})^2 \rangle$ simplifies to a single sum.