Physics 880.06: Problem Set 8

Due Friday, November 22, 2002

Note: each problem is worth 10 pts.

1. *Debye model in two dimensions.* Consider a monatomic lattice in two dimensions, containing N atoms in area S, with Born-von Karman boundary conditions. Assume that the atoms can vibrate only in the plane. In the Debye approximation, we take the vibrational modes to have a dispersion relation $\omega = sk$, where s is the speed of sound, and we approximate the first Brillouin zone by a Debye circle of radius k_D?

   (a). How many polarizations are there per k-vector?

   (b). What is the radius of the Debye circle?

   (c). What is the number of allowed k-points per unit area of k-space?

   (d). Write down the thermal energy U_{th} (i. e. the internal energy in excess of the zero-point energy) in the form of an integral over k-space, assuming the linear dispersion relation given above?

   (e). Define the Debye temperature Θ_D.

   (f). Calculate the specific heat C_V for (i) $T \ll \Theta_D$, and (ii) $T \gg \Theta_D$. In case (i), your result will involve an integral which can be evaluated using the methods of eqs. (23.19) plus Appendix C, eqs. (C.11) to (C.13) of Ashcroft and Mermin; however, I will be satisfied if you just leave your result in terms of this (temperature-independent) integral.

2. Consider the Debye model in three dimensions, but assume that the modes have a dispersion relation $\omega = Dk^a$ where $a \geq 1$. Show in this case that the low-temperature specific heat varies as $T^{3/a}$.

3. In a one-dimensional monatomic lattice, it was shown in class that the vibrational modes satisfy the dispersion relation

$$\omega = \omega_0 \sin(ka/2), \tag{1}$$

where ω_0 is a constant and a is the lattice constant.

Calculate the *density of normal modes* $g(\omega)$ in this case. For a lattice of length L, $g(\omega)d\omega$ is defined as the number of normal modes per unit length between frequency ω and $\omega + d\omega$. Show, in particular, that

$$g(\omega) = \frac{2}{\pi a \sqrt{\omega_0^2 - \omega^2}} \tag{2}$$

4. *Absorption of energy by a driven harmonic oscillator.* In class today, I very briefly described a model for absorption of light by an optical phonon. In this problem, you will work out how much energy is absorbed, by considering a classical model of a simple damped one-dimensional harmonic oscillator of mass m, charge q, spring constant k, and damping parameter Γ, in the presence of an applied electric field $E \exp(-i\omega t)$. You may think of this oscillator as a very crude model of an optical phonon. Assume that the equation of motion for this oscillator is

$$m\ddot{x} = qE \exp(-i\omega t) - m\Gamma\dot{x} - kx. \tag{3}$$

(We use the convention that the physical quantities are the real parts of complex quantities.)

(a). Find the particular solution of the differential equation as a function of the system parameters.

(b). Calculate the time-averaged power P absorbed by the oscillator ($P = \langle F(t)v(t)\rangle$, where $\langle ....\rangle$ denotes a time average, F(t) is the force, and v(t) is the velocity), as a function of frequency. Show that, if Γ is small, P has a sharp peak at $\omega = \omega_0$, where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the oscillator.

(c). The full solution to the differential equation also includes a linear combination of the solutions to the related homogeneous differential equation. Explain why these solutions do not contribute to the time-averaged absorption.