Physics 880K20: Problem Set 5

Due Friday, March 10 by 11:59 PM

1. Prove the *Baker-Campbell-Hausdorf formula*

$$e^{(A+B)\Delta t} = e^{A\Delta t}e^{B\Delta t}e^{-\frac{1}{2}[A,B](\Delta t)^2} + \mathcal{O}((\Delta t)^3). \tag{1}$$

   Here A and B are operators with commutator $[A, B]$, and $\mathcal{O}((\Delta t)^3)$ means a term of order $(\Delta t)^3$.

2. As mentioned in class, one proposed realization of a qubit is the so-called Cooper pair box. This is described by the Hamiltonian

$$H = \frac{U}{2}(n - \bar{n})^2 - J\cos\phi, \tag{2}$$

   where the commutator $[n, \phi] = -i$, and $i = \sqrt{-1}$, and both U and J are positive. The corresponding time-dependent Schrödinger equation is

$$H\psi(\phi, t) = i\hbar\frac{\partial}{\partial t}\psi(\phi, t), \tag{3}$$

   and the energy eigenstates satisfy

$$H\psi_n(\phi) = E_n\psi_n(\phi), \tag{4}$$

   with the boundary condition $\psi_n(\phi + 2\pi) = \psi_n(\phi)$.

   (a). Suppose $J = 0$. Find the eigenstates and eigenvalues of H in this case. Show that for $\bar{n}$ equal to a half-integer,a the lowest two eigenstates are degenerate, and find the eigenvalues and corresponding eigenstates.

   (b). Now suppose that $\bar{n} = \frac{1}{2} + \delta$, where $|\delta| \ll 1$, and also that $J \ll U$. The two lowest eigenstates can then be found approximately by diagonalizing a 2×2 matrix. Find this matrix, and diagonalize it to find the two lowest eigenvalues and corresponding wave functions.

3. (This problem is worth 20 points.) The so-called *phase qubit* also involves a single Josephson junction, but driven by a fixed current rather than a fixed voltage. As mentioned in class, the Hamiltonian of the phase qubit is taken to be

$$H = \frac{U}{2}n^2 - \frac{\hbar I_c}{2e}\cos\phi - \frac{\hbar I}{2e}\phi. \tag{5}$$

Here I_c is the critical current of the Josephson junction, $I < I_c$ is the driving current, and $[n, \phi] = -i$. Thus, a convenient representation of the operator n is $n = -i\frac{\partial}{\partial\phi}$. Usually, this qubit is operated in the parameter range $\hbar I_c/(2e) \gg U$.

(a). Find the value of ϕ (denoted ϕ_{min} in the range $-\pi < \phi < \pi$, such that the potential energy $V(\phi) = -\frac{\hbar I_c}{2e}\cos\phi - \frac{\hbar I}{2e}\phi$ has a relative minimum. (ϕ_{min} is written as an inverse trigonometric function.)

(b). Find the value of ϕ (denoted ϕ_{max}, closest to the value of ϕ_{min} you calculated in (a), such that the potential energy has a relative maximum. Also, find $V_{max} - V_{min}$.

(c). If ϕ has only small oscillations about this minimum, H is approximately the Hamiltonian of a harmonic oscillator. Write down this approximate harmonic oscillator Hamiltonian and find its eigenvalues.

(d). Show that for I only slightly smaller than I_c, this frequency varies approximately as $(1 - I/I_c)^y$ and find the power y. Show also that $V_{max} - V_{min}$ varies approximately as $(1 - I/I_c)^x$ in this range, and find x.

(e). Explain why the small-oscillation approximation is best when $\hbar I_c/(2e) \gg U$.

(f). For $I/I_c = 0.99$, find the smallest value of $\hbar I_c/(2eU)$ such that there are exactly *three* harmonic oscillator eigenvalues with energy below V_{max}. This is roughly the parameter range in which the phase qubit is operated in practice.

4. *Reading of the phase qubit.* In order for the phase qubit to be useful, it must be possible for its state to be read. Typically, the state is read by observing the current generated when the "particle" tunnels out of the metastable well through the barrier. This tunneling rate is different

when the particle is in its ground or its first excited state. In this problem, you will estimate the difference using the WKB approximation.

According to the WKB approximation (see, e. g., Shankar, p. 444), if the particle is in a state $\psi(x)$, where x is the position variable, the tunneling amplitude is proportional to

$$\exp(-\gamma/2) \equiv \exp\left(\frac{i}{\hbar}\int_{x_0}^{x_1}\sqrt{2m(E-V(x))}dx\right). \tag{6}$$

Here E is the energy of the state, $V(x)$ is the potential energy, and x_0 and x_1 are the "classical turning points," i. e., the values of the position to the left and the right of the barrier, such that $V(x_0) = V(x_1) = E$. Since $E - V(x) < 0$ inside the barrier, the square root is purely imaginary, and the exponential factor is negative.

To use the WKB approximation for the Josephson junction problem, you first have to rewrite the WKB approximation so that the Schrödinger equation (which is expressed in terms of a phase variable) is instead written in terms of some position variable. You can do this by writing $\phi = x/r_0$, where r_0 is some arbitrary quantity with dimensions of length. You can then write the WKB approximation in the standard way using the variable x, then finally convert back to the phase variable ϕ.

Show that using this procedure you get for the WKB approximation the following expression:

$$\exp(-\gamma/2) \equiv \exp\left(i\int_{\phi_0}^{\phi_1}\sqrt{\frac{E-V(\phi)}{U}}d\phi\right), \tag{7}$$

where ϕ_0 and ϕ_1 are the two turning points.

Now, from this expression, the tunneling rate is much greater out of the first excited state than out of the ground state, because the barrier is not as wide and its height is not as great.

Make a very rough estimate of the tunneling rate from the lowest two harmonic oscillator eigenstates, using the parameters found in problem 3(f) above together with the WKB approximation. You may use any desired crude approximation to estimate the necessary integral, which probably cannot be done in closed form.