Physics 880K20: Problem Set 4

Due Wednesday, February 22 by 5PM

1. This problem applies to any system with harmonic oscillator modes, including real harmonic oscillators and photons, among others.

   Consider the Hamiltonian $\hbar\omega a^\dagger a$, where $a^\dagger$ and a are the photon creation and annihilation operators. $a^\dagger$ and a satisfy the commutation relations $[a, a^\dagger] = 1$.

   (a). Let $|n\rangle$ be an eigenstate of this Hamiltonian, with eigenvalue E_n. By considering the commutator $[H, a_n^\dagger]$, show that $a^\dagger|n\rangle$ is an eigenstate of H with energy $E_n + \hbar\omega$. Show that if $|n\rangle$ is normalized, then the state $|n+1\rangle = a^\dagger|n\rangle/\sqrt{n+1}$ is also normalized. Hence, show that $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

   (b) Using an analogous method, show that $a|n\rangle = \sqrt{n}|n-1\rangle$.

   (c). Show that $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$, where $|0\rangle$ is the normalized ground state.

   (d). For a harmonic oscillator, if $a = \left(1/\sqrt{m\hbar\omega}\right)(m\omega x + ip)$ and $a^\dagger = \left(1/\sqrt{m\hbar\omega}\right)(m\omega x - ip)$. show that the harmonic oscillator Hamiltonian $H = (1/2)m\omega^2 x^2 + p^2/2m$ can be rewritten as $H = \hbar\omega(a^\dagger a + 1/2)$, where $[a, a^\dagger] = 1$.

2. A laser emits a so-called *coherent state* of a photon field, defined by

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \qquad (1)$$

   where $|n\rangle$ is the normalized state containing n photons. Show that the coherent state is an eigenstate of the photon annihilation operator, i. e., show that $a|\alpha\rangle = \lambda|\alpha\rangle$, for some constant λ, and find λ.

3. Suppose we have two photon states, described by creation and annihilation operators $a^\dagger$, a, and $b^\dagger$, b. Define $a^\dagger a - b^\dagger b = Z$, $a^\dagger b + b^\dagger a = X$, and $i(a^\dagger b - a b^\dagger) = Y$. Show that these operators X, Y, and Z satisfy the usual commutation relations for Pauli operators.

4. Consider a system with two photon modes, as in the previous problem. Define the *Kerr operator* $K = \exp(i\chi L a^\dagger a b^\dagger b)$, where χ and L are constants. Consider the effect of applying K to a state in which the a photons are in a coherent state and the b photons are in a state of definite number. Show that

$$K|\alpha\rangle|n\rangle = |\alpha e^{i\chi L n}\rangle|n\rangle, \tag{2}$$

where $a|\alpha\rangle = \alpha|\alpha\rangle$ and $b^\dagger b|n\rangle = n|n\rangle$. Thus the Kerr operator just shifts the phase of the a photons.

5. In discussing NMR quantum computation in class, we considered two frames of reference: the lab frame, and the rotating frame. If $|\psi(t)\rangle$ is the time-dependent wave function of a spin-1/2 particle in the lab, then by definition the wave function $|\phi(t)\rangle$ in the rotating frame is related to $|\psi(t)\rangle$ by

$$|\phi(t)\rangle = \exp(i\omega t Z/2)|\psi(t)\rangle. \tag{3}$$

$|\psi(t)\rangle$ satisfies the Schrödinger equation $H|\psi(t)\rangle = i\hbar(\partial/\partial t)|\psi(t)\rangle$. In class, we showed that $|\phi(t)\rangle$ satisfies the transformed Schrödinger equation $\tilde{H}|\phi(t)\rangle = i\hbar(\partial/\partial t)|\phi(t)\rangle$, where $\tilde{H} = e^{i\omega t Z/2} H e^{-i\omega t Z/2} - (\omega/2)Z$.

Suppose that $H = (\hbar\omega_0/2)Z + H_{ac}(t)$, and that $\omega = \omega_0$. Find $H_{ac}(t)$ such that $\tilde{H} = g_1(t)X + g_2(t)Y$, where $g_1(t)$ and $g_2(t)$ are specified functions.