Due Wednesday, February 22 by 5PM

1. This problem applies to any system with harmonic oscillator modes, including real harmonic oscillators and photons, among others.

Consider the Hamiltonian $\hbar \omega a^\dagger a$, where $a^\dagger$ and $a$ are the photon creation and annihilation operators. $a^\dagger$ and $a$ satisfy the commutation relations $[a, a^\dagger] = 1$.

(a). Let $|n\rangle$ be an eigenstate of this Hamiltonian, with eigenvalue $E_n$. By considering the commutator $[H, a^\dagger n]$, show that $a^\dagger |n\rangle$ is an eigenstate of $H$ with energy $E_n + \hbar \omega$. Show that if $|n\rangle$ is normalized, then the state $|n + 1\rangle = a^\dagger |n\rangle / \sqrt{n + 1}$ is also normalized. Hence, show that $a^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle$.

(b) Using an analogous method, show that $a |n\rangle = \sqrt{n} |n - 1\rangle$.

(c). Show that $|n\rangle = (a^\dagger)^n |0\rangle$, where $|0\rangle$ is the normalized ground state.

(d). For a harmonic oscillator, if $a = (1/\sqrt{m \hbar \omega}) (m \omega x + ip)$ and $a^\dagger = (1/\sqrt{m \hbar \omega}) (m \omega x - ip)$, show that the harmonic oscillator Hamiltonian $H = (1/2) m \omega^2 x^2 + p^2 / 2m$ can be rewritten as $H = \hbar \omega (a^\dagger a + 1/2)$, where $[a, a^\dagger] = 1$.

2. A laser emits a so-called coherent state of a photon field, defined by

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$  

where $|n\rangle$ is the normalized state containing $n$ photons. Show that the coherent state is an eigenstate of the photon annihilation operator, i.e., show that $a|\alpha\rangle = \lambda |\alpha\rangle$, for some constant $\lambda$, and find $\lambda$.

3. Suppose we have two photon states, described by creation and annihilation operators $a^\dagger$, $a$, and $b^\dagger$, $b$. Define $a^\dagger a - b^\dagger b = Z$, $a^\dagger b + b^\dagger a = X$, and $i(a^\dagger b - ab^\dagger) = Y$. Show that these operators $X$, $Y$, and $Z$ satisfy the usual commutation relations for Pauli operators.
4. Consider a system with two photon modes, as in the previous problem. Define the Kerr operator $K = \exp(i\chi La^\dagger abb^\dagger)$, where $\chi$ and $L$ are constants. Consider the effect of applying $K$ to a state in which the $a$ photons are in a coherent state and the $b$ photons are in a state of definite number. Show that

$$K|\alpha\rangle|n\rangle = |\alpha e^{i\chi Ln}\rangle|n\rangle, \quad (2)$$

where $a|\alpha\rangle = \alpha|\alpha\rangle$ and $b^\dagger b|n\rangle = n|n\rangle$. Thus the Kerr operator just shifts the phase of the $a$ photons.

5. In discussing NMR quantum computation in class, we considered two frames of reference: the lab frame, and the rotating frame. If $|\psi(t)\rangle$ is the time-dependent wave function of a spin-$1/2$ particle in the lab, then by definition the wave function $|\phi(t)\rangle$ in the rotating frame is related to $|\psi(t)\rangle$ by

$$|\phi(t)\rangle = \exp(i\omega t Z/2)|\psi(t)\rangle. \quad (3)$$

$|\psi(t)\rangle$ satisfies the Schrödinger equation $H|\psi(t)\rangle = i\hbar(\partial/\partial t)|\psi(t)\rangle$. In class, we showed that $|\phi(t)\rangle$ satisfies the transformed Schrödinger equation $\tilde{H}|\phi(t)\rangle = i\hbar(\partial/\partial t)|\phi(t)\rangle$, where $\tilde{H} = e^{i\omega t Z/2}He^{-i\omega t Z/2} - (\omega/2)Z$.

Suppose that $H = (\hbar \omega_0/2)Z + H_{ac}(t)$, and that $\omega = \omega_0$. Find $H_{ac}(t)$ such that $\tilde{H} = g_1(t)X + g_2(t)Y$, where $g_1(t)$ and $g_2(t)$ are specified functions.