Physics 880K20, Quantum Computing: Problem Set 3

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Due Wednesday, February 8, 2006

1. [Nielsen and Chuang, (4.1)]. Find the eigenvalues and eigenvectors of the three Pauli matrices, and find the points on the Bloch sphere which correspond to the normalized eigenvectors of the different Pauli matrices.

2. Let x be a real number and A a matrix such that $A^2 = I$. Show that $\exp(iAx) = \cos(x)I + i\sin(x)A$.

3. As stated in class, and discussed in some of the previous homework problems, the three rotation matrices $R_x(\theta)$, $R_y(\theta)$, and $R_z(\theta)$ take the forms:

   $[R_x(\theta)]_{11} = [R_x(\theta)]_{22} = \cos(\theta/2)$, $[R_x(\theta)]_{12} = [R_x(\theta)]_{21} = -i\sin(\theta/2)$;

   $[R_y(\theta)]_{xx} = [R_y(\theta)]_{yy} = \cos(\theta/2)$, $[R_y(\theta)]_{12} = [R_y(\theta)]_{21} = -\sin(\theta/2)$;

   $[R_z(\theta)]_{11} = e^{-i\theta/2}$, $[\mathrm{R}_z(\theta)]_{22} = e^{+i\theta/2}$, $[R_z(\theta)]_{21} = [R_z(\theta)]_{12} = 0$.

   We have also defined the phase gate (denoted S) and the $\pi/8$ gate (denoted T). Both of these are diagonal, with nonzero matrix elements $S_{11} = 1$, $S_{22} = i$; $T_{11} = 1$, $T_{22} = \exp(i\pi/4)$.

   Based on these definitions, and the result of problem 2 if necessary, show that, up to a global phase, the $\pi/8$ gate satisfies $T = R_x(\pi/4)$.

4. [Nielsen and Chuang exercise 4.6]. Suppose a single qubit has a state represented by the Bloch vector λ. Define the rotation operator $R_{\hat{n}}(\theta) \equiv \exp[-i\theta\hat{n}\cdot\sigma/\mathbf{2}]$. Then according to problem 2, this operator may be written

$$R_{\hat{n}}(\theta) = \cos(\theta/2)I - i\sin(\theta/2)(n_xX + n_yY + n_zZ). \qquad (1)$$

Show that the effect of the rotation $R_{\hat{n}}(\theta)$ on the state of the qubit is to rotate it by an angle θ about the $\hat{n}$ axis of the Bloch sphere. This fact explains the rather mysterious-looking factor of two in the definition of the rotation matrices.

To solve the next problem, you need the following theorem [Theorem 4.1 of Nielsen and Chuang] which will be proven in class on Monday:

Theorem: (ZY decomposition for a single qubit.) Suppose U is a unitary operation on a single qubit. Then there exist real numbers α, β, γ, and δ such that

$$U = \exp(i\alpha)R_z(\beta)R_y(\gamma)R_x(\delta). \tag{2}$$

5. Suppose $\hat{m}$ and $\hat{n}$ are non-parallel real unit vectors in three dimensions. Use the preceding theorem to show that an arbitary single qubit unitary operation U may be written

$$U = \exp(i\alpha)R_{\hat{n}}(\beta)R_{\hat{m}}(\gamma)R_{\hat{n}}(\delta) \tag{3}$$

for appropriate (real) choices of α, β, γ, and δ.