Physics 880K20 (Quantum Computing): Problem Set 2.

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Due Monday, January 30 by 5PM.

1. Show that all eigenvalues of a unitary matrix have modulus 1, i. e., that they can be written in the form $\exp(i\theta)$ for real θ.

2. For an n-dimensional complex vector space, one can define an orthonormal set of basis vectors $|i\rangle$, $i = 1, ...n$. A projector operator is defined by

$$P = \sum_{i=1}^{m} |i\rangle\langle i|,$$

where $m \leq n$.

(a) Show that $P^2 = P$.

(b). Show that all eigenvalues of a projector operator P are either zero or 1.

3. Show that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

4. Show that a positive operator is necessarily Hermitian. Hint: First show that an arbitrary operator A can be written as $A = B + iC$, where B and C are Hermitian.

5. Find the square root and natural logarithm of the 2×2 matrix whose elements are $A_{11} = A_{22} = 4$, $A_{12} = A_{21} = 3$.

6. For the Hadamard gate H whose matrix elements are $H_{11} = H_{12} = H_{21} = 1/\sqrt{2}$; $H_{22} = -1/\sqrt{2}$,

(a). Verify that H is unitary and that $H^2 = I$;

(b). Find the eigenvalues and eigenvectors of H.

7. If $\mathbf{v}$ is any three-dimensional vector of length unity, show that $\mathbf{v} \cdot \sigma$ has eigenvalues ± 1, where σ is the ordered triple of Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$.

8. Calculate the probability of getting the result $+1$ for a measurement of $\mathbf{v} \cdot \sigma$, given that the state prior to the measurement was $|0\rangle$.

9. Show that the average value of the observable $X_1 Z_2$ in a two-qubit system measured in the state $(|00\rangle + |11\rangle)/\sqrt{2}$ is zero.