Physics 880K20 (Quantum Computing): Problem Set 1.

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Due Friday, January 20 by 5PM.

1. Supposing a quantum-mechanical state is described by a column vector B. The inner product of two states A and B is denoted by (A, B) = $A^\dagger$ B, where $A^\dagger$ is the Hermitean conjugate of a column vector A, and is a row vector.

   Now consider a linear transformation of this state generated by a matrix U. The transformed state is denoted B' = UB, and the transformed inner product is (A', B').

   Show that the requirement that (A', B') = (A, B) requires that the square matrix U is unitary, i. e., that $U^\dagger$ U = I, where I is the identity matrix. (n × n if A and B have n components).

2. Consider a system described by a time-independent Hamiltonian. The solution of the time-dependent Schrodinger equation for the wave function B is

   $$B(t) = \exp(-iHt/\hbar)B(0).$$

   Show that if H is Hermitean (as it must be for a real physical system) then $\exp(-iHt/\hbar)$ is unitary.

3. **Some properties of Pauli matrices.** Consider the three 2 × 2 matrices X, Y, and Z, The only non-zero components of X are $X_{12} = X_{21} = 1$. The only non-zero components of Y are $Y_{12} = -Y_{21} = -i$. The only non-zero components of Z are $Z_{11} = -Z_{22} = 1$.

   (a) Show that $X^2 = Y^2 = Z^2 = I$, where I is the 2 × 2 unit matrix.

   (b). Show that $[X, Y] = 2iZ$, and cyclic permutations, where [,] denotes a quantum-mechanical commutator ($[A, B] = AB - BA$).

   (c) Show that $R_x = \exp(-i\theta X/2) = \cos(\theta/2)I - iX\sin(\theta/2)$, and similarly for R_y and R_z.

4. Let x be a real number and A be a matrix such that $A^2 = I$. Show that $\exp(iAx) = cos(x)I + i\sin(x)A$. Note: A can be $n \times n$, not just 2×2.

5. As discussed in class, the Hadamard gate is defined by a unitary matrix U with components $U_{11} = U_{12} = U_{21} = 1/\sqrt{2}$, $U_{22} = -1/\sqrt{2}$. Show that the Hadamard gate can be expressed as a product of R_y and R_z rotations and $\exp(i\phi)$ for some ϕ. (Note that $\exp(i\phi)$ is a unitary operator, so the gate is expressible as three successive unitary operations and is thus achievable physically, at least in principle.)