Physics 880.06: Problem Set 2

Due Monday, April 18 by 11:59 P. M.

1. 20 pts.

   (a). Consider a free electron gas of density n electrons per unit volume (in three dimensions). Calculate the electronic density of states N(E) (in states per unit energy per unit volume). Show, in particular that $N(E) = A\sqrt{E}$, and calculate the constant A in terms of fundamental constants (i. e. $\hbar$, electron mass, and electronic charge, if they enter) and the density n.

   (b). Calculate the Fermi energy E_F in terms of n.

   (c). Hence, find $N(E_F)$.

   (d). Find the actual numerical values of these quantities for the metal aluminum, assuming that there are three valence electrons per atom.

   (e). Repeat (a) - (c) for a two-dimensional electron gas, except that n is understood now as the electron density per unit area.

2. (10 pts.)

   We showed in class that the relative wave function of a Cooper pair is proportional to

   $$\sum_{k>k_F} \frac{\cos \mathbf{k} \cdot \mathbf{r}}{2\xi_{\mathbf{k}} + 2E_F - E}, \tag{1}$$

   where (as stated in class) $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - E_F$, $E = 2E_F - 2\hbar\omega_c \exp[-1/(N(0)V)]$, $\hbar\omega_c$ is a suitable cutoff energy, E_F is the Fermi energy, and the sum runs up to $\mathbf{k}$ such that $\epsilon_k = E_F + \hbar\omega_c$. If we define $g_{\mathbf{k}} = 1/[2\xi_{\mathbf{k}} + 2E_F - E]$, then the largest coefficient $g_{\mathbf{k}}$ corresponds to $k = k_F$.

   (a). Estimate the value of $\epsilon_{\mathbf{k}}$ such that $g_{\mathbf{k}}$ equals half its maximum value, and estimate the corresponding value of k. Assume that $\hbar\omega_c$ is much less than E_F. Show that in the weak-coupling regime, this energy satisfies $\epsilon_{\mathbf{k}} = E_F + \Delta$, where Δ is a quantity much less than $\hbar\omega_c$.

(b). Hence, obtain an approximate expression for the *coherence length* (i. e. the size of the Cooper pair in space). Make use of an appropriate uncertainty relation between this size and the spread in k-space of the wave vectors entering the Cooper pair wave function. Make a numerical estimate of this coherence length for aluminum, taking Δ comparable to $k_B T_c$, where T_c is the superconducting transition temperature for aluminum and using a suitable value of the Fermi energy.

3. (20 pts.)

Using the BCS ground state wave function described in class, confirm the statement made in class, that the mean-square fluctuation in electron number in this wave function is

$$< (N - \bar{N})^2 >= 4 \sum_{\mathbf{k}} u_{\mathbf{k}}^2 v_{\mathbf{k}}^2. \tag{2}$$

where $\bar{N}$ is the expectation value of the number of electrons in this wave function, given by $\bar{N} = 2 \sum_{\mathbf{k}} v_{\mathbf{k}}^2$. (You don't have to prove this last equality; just assume it.)

You may take the coefficients $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ to be real.

Hint: first show that

$$< \Psi_G | n_{\mathbf{k}\sigma} n_{\mathbf{k}'\sigma'} | \Psi_G >=< \Psi_G | n_{\mathbf{k}\sigma} | \Psi_G >< \Psi_G | n_{\mathbf{k}'\sigma'} | \Psi_G > \tag{3}$$

for $\mathbf{k} \neq \mathbf{k}'$, and

$$< \Psi_G | n_{\mathbf{k}\sigma} n_{\mathbf{k}'\sigma'} | \Psi_G >= \langle \Psi_G | n_{\mathbf{k}\sigma} | \Psi_G \rangle \delta_{\sigma\sigma'}. \tag{4}$$

for $\mathbf{k} = \mathbf{k}'$. (Here, σ is a spin index and $|\Psi_G\rangle$ denotes the ground state. With these results, you should be able to show that $\langle (N - \bar{N})^2 \rangle$ simplifies to a single sum.

Recall that the BCS ground state wave function is

$$|\Psi_G\rangle = \Pi (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k},\uparrow}^{\dagger} c_{-\mathbf{k},\downarrow}^{\dagger}) |0\rangle, \tag{5}$$

where $|0\rangle$ is the vacuum state, and Π denotes a product over all $\mathbf{k}$'s. Also recall that $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$.