Physics 848: Problem Set 3

Due Tuesday, November 8 at 11:59 PM

Note: each problem is worth 10 points unless otherwise stated.

1. In class it was shown that the Ornstein-Zernike correlation function $\Gamma(\mathbf{r}-\mathbf{r}')$ was given by

$$\Gamma(\rho) = \int \frac{1}{a/2 + Ck^2} e^{i\mathbf{k}\cdot\rho} d^3k \tag{1}$$

in three dimensions. (Here $\rho = \mathbf{r} - \mathbf{r}'$.)

Show that this integral is given by

$$\Gamma(\rho) = \frac{K}{\rho}\exp(-\rho/\xi), \tag{2}$$

where K is a constant and $\xi = \sqrt{2C/a}$ is the correlation length. [Thus, if $a \propto (T - T_c)$, $\xi \propto (T - T_c)^{-1/2}$.]

Hint: first introduce spherical coordinates such that the polar angle θ is the angle between $\mathbf{k}$ and ρ. Then carry out the angular integrals. The remaining integral can be arranged so that it can be done as a contour integral involving simple poles.

2. **Critical exponents for the Van der Waals equation of state.** As given in class, the Van der Waals equation of state may be written

$$\left(P + \frac{3}{V^2}\right)\left(V - \frac{1}{3}\right) = \frac{8}{3}T. \tag{3}$$

Here $P = p/p_c$, $V = v/v_c$, and $T = t/t_c$, where p, v, and t are the pressure, specific volume (volume per atom) and temperature, and p_c, v_c, and t_c are the corresponding values of these quantities at the critical point.

(a). Calculate the first and second derivatives of pressure with respect to volume, i. e. $(\partial p/\partial v)_T$ and $(\partial^2 p/\partial v^2)_T$, as functions of v and T.

(b). Explain why $-(\partial p/\partial v)_T$ must be non-negative in order for the system to be stable.

(c). At the liquid-gas critical point, $-(\partial p/\partial v)_T = 0$. In addition, $(\partial^2 p/\partial v^2)_T = 0$, because the critical point should be an extremum of $(\partial p/\partial v)_T$. From these two conditions, show that the critical point occurs at $P = 1$, $V = 1$, and $T = 1$.

(d). Calculate the isothermal compressibility κ_T for T slightly greater than 1, and show that it is proportional to $(T-1)^{-1}$. Explain why κ_T is analogous to the magnetic susceptibility. From this result, you may conclude that the critical exponent $\gamma = 1$.

(e). Show that exactly at $T = 1$, $|V-1| \propto |P-1|^{1/3}$ (to lowest order in $|P-1|$. Since $|V-1|$ is analogous to the order parameter of the liquid-gas phase transition, and $|P-1|$ is analogous to the applied magnetic field, this result means that the critical exponent $\delta = 3$.

3. **Tonks Gas.** Consider a one-dimensional gas of particles of length a confined to a strip of length L. The particles interact through a potential

$$U(x_i - x_j) = \infty \quad \text{for } |\mathrm{x_i - x_j}| < \mathrm{a} \tag{4}$$

and

$$U(x_j - x_j) = 0 \quad \text{for } |\mathrm{x_i - x_j}| > \mathrm{a.} \tag{5}$$

Calculate the partition function and equation of state exactly.

4. **Bloch wall.** In class, I introduced the Ginzburg-Landau free energy $G = \int [\frac{a}{2}m^2 + \frac{b}{4}m^4 + C|\nabla m|^2] d^3x$, where m is a scalar order parameter (say the magnetization), and a, b, and C are coefficients.

Suppose that $a < 0$, $b > 0$, and $C > 0$. Then, as shown in class, the values of m that minimize the free energy for a uniform system are $m = \pm\sqrt{-a/b}$. Now assume that we have a magnetic material such that $m \to \pm\sqrt{-a/b}$ as $x \to \pm\infty$. The magnetization m must be a function of x in order to go from one limit to the other. (For the given boundary conditions, m will not depend on y and z.) The transition between the two regions, one with up and one with down magnetization, is called a Bloch wall, or a domain wall.

(a) Show that the m(x) which minimizes the free energy for these boundary conditions satisfies the nonlinear differential equation

$$am + bm^3 - C\frac{d^2m}{dx^2} = 0. \tag{6}$$

(b). Show, by direct substitution, that a solution to this equation which satisfies the boundary conditions is

$$m(x) = A\tanh\left(Bx\right), \tag{7}$$

and find A and B in terms of a, b, and C.

(c). Calculate the extra free energy associated with the Bloch wall.