Physics 848: Problem Set I

Due Thursday, October 6 at 11:59 PM

Note: each problem is worth 10 points unless otherwise stated.

1. Use the method presented in class to obtain the exact partition of the one-dimensional antiferromagnetic Ising model in a magnetic field. Assume that the number of spins N is very large, but is an even number. Also compute the magnetic susceptibility and the specific heat at constant volume.

2. An alternative way of obtaining the mean-field critical temperature of the Ising model is the following. Consider the ferromagnetic Ising model with z nearest neighbors. In an applied magnetic field B. the total effective field felt by a given spin is the sum of two parts: the applied field B, and the effective field due to the nearest neighbors. This second field to be $zJ\langle S\rangle$, where $\langle S\rangle$ is the expectation value of the spin on one of the z nearest neighbors.

   (a). Use this approximation to obtain a self-consistent equation for the spin $\langle S\rangle$.

   (b). Hence, obtain T_c as the highest temperature for which this equation has a nonzero solution for $\langle S\rangle$ in the limit of zero applied magnetic field. Show that this solution is the same as that found in class using the Bragg-Williams approach.

3. Consider the "infinite range Ising model," where the coupling $J_{ij} = J$ for all distinct pairs of spins, with no restriction to nearest neighbors. Thus, the Hamiltonian for a system of volume V is

$$H = -B\sum_i S_i - \frac{J_0}{2}\sum_{ij} S_iS_j. \tag{1}$$

   (a). Explain why this model only makes sense if $J_0 = J/N$, where N is the number of spins in the system.

(b), Prove that

$$\exp\left(\frac{ax^2}{2N}\right) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi/(Na)}} \exp\left(-\frac{Na}{2}y^2 + axy\right), \tag{2}$$

provided that Re a is greater than 0.

(c). Hence, show that the partition function is given by

$$Q = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi k_B T/(NJ)}} e^{-N\beta L}, \tag{3}$$

where

$$L = \frac{Jy^2}{2} - \frac{1}{\beta} ln[2cosh(\beta(B + Jy))]. \tag{4}$$

Here $\beta = 1/(k_B T)$, where T is the absolute temperature.