Physics 847: Problem Set 7

Due Thursday, May 27, 2010 at 11:59 P. M.

Each problem is worth 10 points unless specified otherwise.

1. Consider the spin-1/2 Heisenberg model in an applied magnetic field \( \mathbf{B} = B\hat{z} \). The Hamiltonian may be written

\[
H = -\frac{J}{\hbar^2} \sum_{\langle ij \rangle} \left[ \frac{1}{2} (S_i^+ S_j^- + S_i^- S_j^+) + S_{iz} S_{jz} \right] - \frac{g e B}{2mc} \sum_i S_{iz}.
\]

(a). What is the ground state energy and spin configuration? If \( B \neq 0 \), is this state degenerate?

(b). Calculate the spin wave spectrum for this Hamiltonian, following the approach used in class. Show, in particular, that the spectrum has a gap, i.e., the lowest spin wave excitation has an energy of \( \Delta \neq 0 \), and find \( \Delta \). What is the temperature dependence of the spin wave specific heat in this case at low temperatures \( k_B T \ll \Delta \)?

2. Repeat problem 1 for the anisotropic Heisenberg model at zero magnetic field,

\[
H = -\frac{1}{\hbar^2} \sum_{\langle ij \rangle} \left[ \frac{J_\|}{2} (S_i^+ S_j^- + S_i^- S_j^+) + J_\perp S_{iz} S_{jz} \right],
\]

for the case \( J_\| > J_\perp \).

3. Consider the commutator \([S_{k,-}, S_{k,+}]\), where

\[
S_{k, \pm} = \frac{1}{\sqrt{N}} \sum_{\ell} \exp(\pm i k \cdot R_\ell) S_{\ell, \pm},
\]

and the spin operators all refer to spin-1/2 particles. Show that, if this commutator operates on a state in which all the spins are parallel and equal to \(-\hbar/2\), then \([S_{k,-}, S_{k,+}] = \hbar\). Thus, \( S_{k,-}/\sqrt{\hbar} \) and \( S_{k,+}/\sqrt{\hbar} \) have
the same commutation relations as the raising and lowering operators \( a \) and \( a^\dagger \) for harmonic oscillators. This shows that spin wave states, like harmonic oscillator states, can be treated as bosons with a zero chemical potential.

4. At low temperatures, the magnetization of the ferromagnetic spin-1/2 Heisenberg model satisfies \( \mathbf{M} = M\hat{z} \), where

\[
M(T) = M(0) - C \sum_k \langle n_k \rangle. \tag{4}
\]

Here \( C \) is a constant and \( \langle n_k \rangle \) is the expected number of spin waves (magnons) in state \( k \) at temperature \( T \). You do not have to derive this result; just assume it.

(a). Show that this result implies that for a simple cubic lattice at low temperatures,

\[
M(T) = M(0) - KT^s \tag{5}
\]

where \( s = 3/2 \).

(b). Show that, for a SQUARE lattice at any non-zero temperature, \( \sum_k \langle n_k \rangle \) diverges. Hence, the magnetization vanishes at any finite temperature.

5. Calculate the second virial coefficient for a gas described by the pair potential

\[
V(r) = \infty \tag{6}
\]

for \( r < a \);

\[
V(r) = -V_0 \tag{7}
\]

for \( a < r < b \);

\[
V(r) = 0 \tag{8}
\]

for \( r > b \). Take \( V_0 > 0 \) and \( b > a \).