Physics 847: Problem Set 6

Due Thursday, May 20, 2010 at 11:59 P. M.

Each problem is worth 10 points unless specified otherwise.

1. Pathria, Problem 11.7

2. (25 pts.) Consider a one-dimensional chain of particles connected by springs. Let the mass of each particle be \( m \), and let each spring constant be \( K \). Suppose that the equilibrium length of each spring is \( a \), and let the position of the \( n^{th} \) mass be denoted \( u_n + a \). We assume that there are a total of \( N \) masses with periodic boundary conditions, so that \( u_{N+1} = u_1 \). The masses are constrained to vibrate only along the chain.

Thus, the Hamiltonian of the system may be written

\[
H = \sum_{n=1}^{N} \left[ \frac{p_n^2}{2m} + \frac{K}{2} (u_{n+1} - u_n)^2 \right].
\]  

(a). Write down the set of \( N \) coupled second-order equations of motion for the \( u_n \)'s, either by solving Hamilton’s equations or by using Newton’s laws directly.

(b). Try a solution of the form \( u_n = u_0 \cos(nka - \omega t) \), where \( u_0 \) is a constant. Write down the relation between \( \omega \) and \( k \) such that this form satisfies the equation of motion.

(c). Show that modes with wave vector \( k \) and \( k + 2\pi/a \) are actually equivalent (i.e., involve the same displacements of the particles). Hence, one only needs to consider modes between \( k = -\pi/a \) and \( k = +\pi/a \).

(d). What are the allowed values of \( k \), given the periodic boundary condition? Show that there are a total of \( N \) allowed values of \( k \) if there are \( N \) particles.

(e). Write down the internal energy \( U \) as a function of temperature \( T \) corresponding to these modes in the limit of very large \( N \), taking
into account the fact that each mode is a quantized lattice vibration (as discussed in class). \( U \) will have the form of an integral (which you should not evaluate in general).

(f). Calculate the zero point energy \( U(T = 0) \).

(g). Find the leading non-vanishing term in the specific heat \( C_V \) at low \( T \), and define what is meant by “low” in this context.

3. Pathria, Problem (12.6).

4. In the Heisenberg model of ferromagnetism, there are excitations at low temperatures called “spin waves” (sometimes called “magnons”). For a simple cubic lattice of spins with lattice constant \( a \), these spin waves have a frequency and wave number which are related by

\[
\omega(k) = \frac{2D}{a^2} [3 - \cos(k_x a) - \cos(k_y a) - \cos(k_z a)], \tag{2}
\]

where \( D \) is called the spin-wave stiffness constant. These spin waves can be considered to be bosons with a zero chemical potential, like lattice vibrations. For a simple cubic lattice, each component of \( k \) must lie in the range \(-\pi/a < k_x, k_y, k_z < \pi/a\).

Show that, at sufficiently low temperatures, the specific heat \( C_V \) due to spin waves satisfies \( C_V = K_1 T^s \), and find \( K_1 \) and \( s \).