Interacting Systems

E.g. ferromagnet (such as Fe, Co, Ni, Gd)

Non-zero magnetization even with zero applied magnetic field

\[ |M| \]

\[ (B=0) \]

Called the Curie temperature \( T_c \)

\( T_c \approx 800-1000K \) for Fe

Magnetic susceptibility

\[ \chi_m = \left( \frac{\partial M}{\partial B} \right) \]

\[ B=0 \]

\[ \sim \frac{1}{T-T_c} \] for \( T>T_c \)

I will discuss specific heat and other properties shortly.

Cannot treat ferromagnets as a collection of independent spins
Recall from last quarter that if we have a collection of independent spins

\[ X = \frac{1}{2} \quad \text{(for classical spins)} \]

and we just showed that

\[ X \text{ is T independent for electron gas (or Fermi gas).} \]

Where does magnetic moment come from?

E.g., Fe, \( Z = 26 \). Outer electrons

There is a net spin per atom, and the spins interact in such a way as to favor an overall magnetic moment which is non-zero.

Question: What is a reasonable hamiltonian to describe the ferromagnetic-to-non-ferromagnetic transition in Fe (and other ferromagnets)?
Interacting Systems: the Ising model:

Lattice of spins:

\[ \uparrow \quad \downarrow \quad \uparrow \quad \downarrow \quad \uparrow \quad \downarrow \quad \ldots \]

Spin is either up or down \( S_i = \pm 1 \)

We assume

\[ H = - \sum_{(ij)} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - \sum_i S_i B \]

\( B \) equals external magnetic field

Spins on a lattice.

Second term: \( - \sum_i S_i B \) is like the term previously discussed, which produces the Curie law in spin-\( \frac{1}{2} \) particles

First term: where does it come from?
simplest case: \( J_{ij} = 0 \).

Then

\[
H = - B \sum_i S_i
\]

What is \( \langle S_i^z \rangle \)? As we showed last quarter,

\[
\langle S_i^z \rangle = \frac{1}{e^{B/k_BT} + e^{-B/k_BT}} + \frac{1}{e^{B/k_BT} + e^{-B/k_BT}}
\]

\[
= \tanh \left( \frac{B}{k_BT} \right)
\]

\[
M = \frac{\sum_i \langle S_i^z \rangle}{V} = \frac{N}{V} \tanh \left( \frac{B}{k_BT} \right)
\]

For small \( B \), \( \tanh \frac{B}{k_BT} \approx \frac{B}{k_BT} \) so

\[
M \approx \frac{N}{V} \frac{B}{k_BT}
\]

\[
\chi_M = \left( \frac{\partial M}{\partial B} \right)_{B=0} = \frac{N}{V} \frac{1}{k_BT}
\]

Curie Law:

There is no temperature at which there is any spontaneous magnetization.
"Nearest neighbor ferromagnetic Ising model"

\[ H = - \sum_{i,j} J_{ij} S_i S_j - \sum_{i} B_i S_i \]

\[ \langle i,j \rangle \]

\[ S_i = \pm 1 \]

\[ J_{ij} = J \quad \langle i,j \rangle \text{ nearest neighbors} \]

\[ = 0 \quad \text{otherwise} \]

\[ \Rightarrow \text{"ferromagnetic Ising model."} \]

Can think of \( S_i \)'s as "spins"

Quantum-mechanical origin of Ising interaction is called "exchange" energy — will discuss later (maybe)

Simplest example of \( S_i \)'s interacting system (or one of the simplest examples)
Some elementary properties

at $B = 0$.

Ground state: all spins aligned

$$ U = -\frac{N J \mathcal{Z}}{2}, \quad M = \frac{N}{V} \mathcal{Z} \langle S \rangle = \frac{N}{V} (\pm 1) $$

$\mathcal{Z} = \# \text{of nearest neighbors (increased)}$

At high $T$: entropy is favored by having

random config. of spin $\Rightarrow$ no order:

We know that at fixed $V, T$ we want to minimize

Need to calculate

$$ q_N(V,T) = \text{Tr} e^{-\beta H} $$

$$ e^{-\beta H(\{S_i\})} = \mathcal{Z} \cdots \mathcal{Z} e^{-\beta H(\{S_i\})} $$

$$ = \mathcal{Z} \cdots \mathcal{Z} e^{-\beta \mathcal{H}} $$

$$ 3 \mathcal{S}_i = \pm 1 $$

Obviously a toughie for real system.

Mean-field theory we

know "Mean-field" approximation with $B = 0$.

Want to minimize $F = U - TB_s S$

$$ U = -J \sum_{\langle ij \rangle} \langle s_i s_j \rangle \quad S = \text{entropy} $$
Introduce $\bar{S}_i = \langle S_i \rangle$

Expect $\bar{S}_i = \bar{S}$ indep of site

$\bar{S} = \"order parameter\" = magnetic moment per site$

$$U = -J \sum_{\langle ij \rangle} \langle S_i S_j \rangle = -J \sum_{\langle ij \rangle} \langle S_i \rangle \langle S_j \rangle$$

$$= -\frac{NJ^2}{2} \bar{S}^2$$

How about entropy?

$$S = k_B \ln \left( \text{# of configurations} \right)$$

# of up spins $= N_\uparrow$

# of down spins $= N_\downarrow$

$$N_\uparrow + N_\downarrow = N$$

$$N_\uparrow - N_\downarrow = N \bar{S} = 2N_\uparrow - N$$

$$N_\uparrow = \frac{N}{2} \left( 1 + \bar{S} \right)$$

$$N_\downarrow = \frac{N}{2} \left( 1 - \bar{S} \right)$$

$$S = k_B \ln \left( \binom{N}{N_\uparrow} \right)$$
\[ \begin{align*}
\sum_{N} & = k_B \ln \left[ \frac{N!}{\left[ \frac{N}{2} (1+3) \right]! \left[ \frac{N}{2} (1-5) \right]!} \right] \\
& = N k_B \ln N - N k_B - \frac{N}{2} (1+5) k_B \ln \left( \frac{N}{2} (1+5) \right) \\
& \quad + \frac{N}{2} k_B (1+5) - \frac{N}{2} (1-5) k_B \ln \left( \frac{N}{2} (1-5) \right) \\
& \quad + \frac{N}{2} k_B (1-5) \\
& = N k_B \left\{ \ln N - \frac{1}{2} (1+5) \ln \left( \frac{N}{2} (1+5) \right) - \frac{1}{2} (1-5) \ln \left( \frac{N}{2} (1-5) \right) \right\} \\
& = \frac{N k_B}{2} \left\{ \frac{1+5}{2} \ln \frac{1+5}{2} + \frac{1-5}{2} \ln \frac{1-5}{2} \right\} \\
\end{align*} \]

So therefore, we have

\[ F_{MF} = -\frac{N J S^2}{2} + N k_B T \left\{ \frac{1+5}{2} \ln \frac{1+5}{2} + \frac{1-5}{2} \ln \frac{1-5}{2} \right\} \]

\[ T > T_c = \frac{N k_B T_{\text{em}}}{2} \]

\[ T = T_c \quad \left( \frac{\partial F}{\partial T} \right)_T = 0 \]

\[ T < T_{c1} \]
Right at $T_c$:

$$\left( \frac{\partial^2 F_{MF}}{\partial \overline{s}^2} \right)_{\overline{s}=0} = 0$$

(always true at $\overline{s}=0$)
(by symmetry)

and

$$\left( \frac{\partial^2 F_{MF}}{\partial \overline{s}^2} \right)_{\overline{s}=0} = 0$$

\[
\frac{\partial^2 F_{MF}}{\partial \overline{s}^2} = -N J z + N k_B T \left[ \frac{1}{1+\overline{s}} + \frac{1}{1-\overline{s}} \right]
\]

\[
= -N J z + N k_B T \quad \text{at } \overline{s}=0
\]

\[
= 0 \quad \text{when } k_B T = J z = k_B T_c
\]

$z$ = # of nearest neighbors:

Two equally good solutions

$T_c = \frac{2J}{k_B}$

Low $T$: energy wins
High $T$: entropy wins.
"Exact" (or rather, exact):

- $z = 6$, 3D: There is a transition, but at somewhat lower $T$
- $z = 4$, 2D: Also, there is a transition
- $z = 2$, 1D: No transition at all

Some properties near $T_c$:

$$F_{MF}(S, T) : \text{expand in power series near } T = T_c.$$  

$$\ln(1 + s) = S - \frac{s^2}{2} + \frac{s^3}{3} - \ldots$$

$$F_{MF} \propto -NJz \frac{S^2}{2} + N k_B T \left\{ \frac{1}{2} \left( S - \frac{S^2}{2} + \frac{S^3}{3} - \frac{S^4}{4} \right) \right. $$

$$\left. - \frac{1 + S}{2} \ln \frac{S}{2} + \frac{1 - S}{2} \left( -S - \frac{S^2}{2} - \frac{S^3}{3} - \frac{S^4}{4} \right) \right.$$

$$\left. - \frac{1 - S}{2} \ln \frac{2}{S} \right\}$$

$$\approx -NJz \frac{S^2}{2} + N k_B T \left\{ \frac{1}{2} \left( -S^2 - \frac{S^4}{2} \right) + \frac{S^2}{2} + \frac{S^4}{3} - \ln 2 \right\}$$

$$= NS^2 \left( \frac{k_B T + J}{2} \right) + \frac{N k_B T}{S^4} \left( \frac{S^4}{12} - \ln 2 \right)$$
or \[ F_{\text{MF}} = \frac{Nk_B}{2} \langle S \rangle^2 (T - T_c) + \frac{Nk_B T}{12} \langle S \rangle^4 - Nk_B T \log 2 \]

\[ \langle S \rangle(T) \text{ for } T < T_c: \]

\[ \frac{1}{N} \frac{\partial^2 F}{\partial S^2} = \kappa_B \frac{1}{3} (T - T_c) + \kappa_B \frac{1}{3} \langle S \rangle^2 = 0 \]

\[ \langle S \rangle^2 = 3 \left( \frac{T_c - T}{T} \right) \quad \langle S \rangle \sim \sqrt{3} \left( \frac{T_c - T}{T} \right) \quad \text{near } T_c: \]

So \[ \langle S \rangle \propto \left( \frac{T_c - T}{T} \right)^{1/2} \]

\[ (2). \quad C_V = \frac{\partial^2 F}{\partial T^2} \]

\[ T > T_c \quad C_V = 0 \]

\[ T < T_c \quad \langle S \rangle^2 \sim \frac{3}{T} \frac{T_c - T}{T_c} \]

\[ F = \frac{3Nk_B}{2} \frac{(T_c - T)^2}{T_c^2} + \frac{Nk_B T_c}{12} \left( \frac{T_c - T}{T_c} \right)^2 \]

\[ \chi = \frac{3}{4} \frac{Nk_B T_c}{T_c^2} \left( \frac{T_c - T}{T_c} \right)^{1/2} + \Theta \left( \frac{T_c - T}{T_c} \right)^3 \]
\[ CV = + \frac{3}{4} Nk_B T_c \]

\[ T \to T_c^- \]

So \( S \sim (T_c - T)^{1/2} \)

and \( CV \) has discontinuity.

\[ \beta \sim \frac{1}{2} \text{ for } d \geq 4 \]

Exact:

\[ M \sim (T_c - T)^\beta \]

\[ \beta \sim \frac{1}{3} \text{ in } 3d \]

\[ \sim \frac{1}{8} \text{ in } 2d \]

\[ CV \sim (T_c - T)^{-\alpha} \]

\[ \alpha \sim \text{small in } 3d \]

\[ \sim \ln(T_c - T) \text{ in } 2d. \]