Suppose $\hat{p} = p \hat{x}$ (11.1).

Then we can work out $\vec{B} = \vec{\nabla} \times \vec{A}$; result is

$$\vec{B}(x, y, z, t) = \frac{3(x-vt)(y \hat{z} - z \hat{y}) \rho v}{\sqrt{(x-xt)^2 + y^2 + z^2}^{5/2}}$$

So moving electric dipole generates a magnetic field.

---

Mathematical properties of Lorentz transform:

First, recall notations in 3-space.

Let us start with a vector $x$ (actually a contravariant vector $x$). It transforms according to the matrix relation

$$x' = Ax$$

The norm of the 4-vector is defined by

$$\gamma g x = (x^1, x^2, x^3, x^4) \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}$$
We can write this as
\[ g = g_{\mu\nu} = \text{metric tensor} \]

A Lorentz transformation is a transformation
\[ x' = Ax \]

such that \( x' g x' = x g x \)

What does this say about the matrix \( A \)?

Well, \( x' = Ax \), so \( x' = \tilde{x} A \)

Therefore \( \tilde{x} A g A x = \tilde{x} g x \) for any \( x \).

Therefore \[ \tilde{A} g A = g \]

Take \( \det \) of both sides
\[
\det (\tilde{A} g A) = \det (g)
\]

\[ = \det (\tilde{A}) \det (g) \det (A) \]

\[ = \det (g) [\det A]^2 = \det (g) \]

\[ \Rightarrow [\det A]^2 = 1 \]

\[ \det A = \pm 1 \]

Proper Lorentz transforms: \( \det A = +1 \)

Improper Lorentz transform: \( \det A = -1 \)
Also, there seem to be 16 eqns. for the 16 components of $A$.

But actually, only 10 eqns. because the matrix $\hat{g} A \hat{g}$ is symmetric:

\[ (\hat{g} A \hat{g}) = \hat{g} \hat{g} A = \hat{g} \hat{g} A \quad \text{since} \quad g = \hat{g}. \]

So 6 indepent parameters.

For proper Lorentz transforms, there are three angles specifying rotation and three velocities (components of $\hat{B}$).

*Improper Lorentz transform has $\det(A) = -1$.*

But even if $\det(A) = +1$ can be an improper Lorentz transf.

E.g. $A = \hat{g}$ (space inversion)
with $\det A = -1$

$A = -I$ (space and time inversion, but $\det A = 1.$)
Construction of $A$ for proper Lorentz transforms

Assume

$$ A = e^L $$

$$ \det A = \det(e^L) $$

Supp. $L = 4 \times 4$ matrix. It can be shown mathematically that $\det(e^L) = e^{(\text{Tr}L)}$

If $\text{Tr}L \neq 0$ then $\det A \neq 1$ so for proper Lorentz transform, $L$ is traceless

Also

$$ \hat{A} g A = g $$

Left multiply by $g$ to get

$$ g \hat{A} g A = g^2 = I \quad \Rightarrow \quad g \hat{A} g = A^{-1} $$

$$ g(e^L) g = A^{-1} $$

$$ = g e^L g \quad \text{since} \quad e^L = 1 + L + \frac{L^2}{2} + \frac{L^3}{3!} $$

$$ = e g \hat{L} g \quad \text{since} $$

$$ e g \hat{L} g = 1 + g \hat{L} g + \frac{1}{2} g \hat{L} g g \hat{L} g + \frac{1}{3!} (g \hat{L} g)^3 + ..$$

$$ = g \left[ 1 + \hat{L} + \frac{1}{2} \hat{L}^2 + \frac{1}{3!} \hat{L}^3 + .. \right] g $$

$$ = g \hat{L} g $$
So \( e^{g\hat{g}} = A^{-1} = e^{-L} \)

or \( g\hat{g} = -L \) or \( g\hat{g} = -Lg = \hat{g}L = L\hat{g} = -Lg \)

So \( Lg \) is antisymmetric

\[
Lg = \begin{pmatrix}
L_{00} & L_{01} & L_{02} & L_{03} \\
L_{10} & L_{11} & L_{12} & L_{13} \\
L_{20} & L_{21} & L_{22} & L_{23} \\
L_{30} & L_{31} & L_{32} & L_{33}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-1 & 0 \\
0 & -1 \\
0 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
L_{00} & -L_{01} & -L_{02} & -L_{03} \\
L_{10} & -L_{11} & -L_{12} & -L_{13} \\
L_{20} & -L_{21} & -L_{22} & -L_{23} \\
L_{30} & -L_{31} & -L_{32} & -L_{33}
\end{pmatrix}
\]

is antisymm.

So \( L_{10} = L_{01} \), \( L_{12} = -L_{21} \), \( L_{20} = L_{02} \), \( L_{13} = -L_{31} \), \( L_{30} = L_{03} \), \( L_{23} = -L_{32} \)

and \( L = \begin{pmatrix}
0 & L_{01} & L_{02} & L_{03} \\
L_{01} & 0 & L_{12} & L_{13} \\
L_{02} & -L_{12} & 0 & L_{23} \\
L_{03} & -L_{13} & -L_{23} & 0
\end{pmatrix} \)

Six params.

Specifying Lorentz transformation
Can write these as:

$$L = -\hat{\omega} \cdot \hat{S} - \hat{S} \cdot \hat{K}$$

$$\hat{S}_1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{S}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\hat{S}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{K}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{K}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{K}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\omega = \text{rotation in 3-space through 3 angles}$$

$$\hat{S} = "\text{boost vector}"$$
We have \( \hat{A} \hat{g} A = g \).

Therefore
\[
\hat{g} \hat{A} \hat{g} A = g^2 = I
\]
or
\[
\hat{g} \hat{A} g = A^{-1}
\]

Now \( A = e^L = I + L + \frac{1}{2} L^2 + \frac{1}{6} L^3 + \ldots \)
\[
\hat{A} = \hat{I} + \hat{L} + \frac{1}{2} \hat{L}^2 + \frac{1}{6} \hat{L}^3 + \ldots
\]

But \( \hat{L}^2 = \hat{L} \hat{L} = \hat{L}^2 \), etc.

So \( \hat{A} = e^{\hat{L}} \)

Also \( A^{-1} = e^{-L} \)

So \( \hat{g} \hat{A} e^L \hat{g} = e^{-L} = e^{\hat{L}} \hat{g} \hat{g} e^{\hat{L}} \hat{g} = e^{\hat{L}} \hat{g} \hat{g} e^{\hat{L}} \hat{g} \)

Proof
\[
e^{\hat{L}} \hat{g} \hat{g} = I + g \hat{L} \hat{g} + g \hat{L} \hat{g} g \hat{L} \hat{g} + \frac{(g \hat{L} \hat{g})^2}{2} + \frac{(g \hat{L} \hat{g})^3}{6} + \ldots
\]

\[
= I + g \hat{L} \hat{g} + g \hat{L}^2 \hat{g} + g \hat{L}^3 \hat{g} + \ldots
\]

\[
= g (I + \hat{L} + \frac{1}{2} \hat{L}^2 + \frac{1}{6} \hat{L}^3 + \ldots) = g e^L \hat{g}
\]

Thus \( e^{\hat{L}} \hat{g} = e^L \hat{g} \)

or \( g \hat{L} \hat{g} = e^L \hat{g} \) or \( \hat{g} \hat{L} = e^{-L} \hat{g} - \hat{L} \hat{g} \)
but our $\tilde{L} \mathbf{g} = \tilde{L} \mathbf{g} = \mathbf{g} L$

So $\tilde{L} L = -L \tilde{L}$

Thus the matrix $L \tilde{L}$ is antisymmetric

See my previous notes

$L = -\vec{\omega} \times \tilde{L} - \frac{2}{3} \tilde{R} L$

Find that $[S_x, S_y] = S_z \mathbb{1}$ and cyclic

where $[S_x, S_y] = S_x S_y - S_y S_x$

Also $[K_x, K_y] = -K_z \mathbb{1}$ and cyclic

E.g. $\vec{\omega} = 0 \quad \tilde{L} = \tilde{g} \tilde{g} \tilde{g}$

$L = -\tilde{g} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$e^{-L} = \mathbb{1} \quad \text{Well,} \quad K^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Thus

$e^{L} = e^{\frac{1}{2} \tilde{g} K_1} = \mathbb{1} - \tilde{g} K_1 + \frac{1}{2!} \tilde{g}^2 K_1^2 - \frac{1}{6!} \tilde{g}^3 K_1^3$
Just consider $2 \times 2$ part of matrix:

$$
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 + \frac{1}{2!} \beta^2 + \frac{1}{4!} \beta^4 + \frac{1}{6!} \beta^6 + .. \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2!} + \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + .. \\
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\cosh \beta & -\sinh \beta & 0 \\
-\sinh \beta & \cosh \beta & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
$$

$cosh^2 - sinh^2 = 1$

Let $\sinh \beta = \beta \alpha$

Then $cosh \beta = \sqrt{1 + sinh^2 \beta} = \beta \alpha \sqrt{1 + \beta^2 \alpha^2}$

$$
= \sqrt{1 + \frac{\beta^2}{1-\beta^2}} = \frac{1}{\alpha}
$$

So $A = \begin{pmatrix}
\frac{1}{\beta \alpha} & -\beta \alpha & 0 \\
-\beta \alpha & \frac{1}{\beta \alpha} & 0 \\
0 & 0 & 1 \\
\end{pmatrix}$

$\beta = sinh^{-1}(\beta \alpha)$ is called the "boost"
More on equations of motion in an EM field

I already wrote down

\[ m \frac{d U^x}{d \tau} = q \frac{F^x}{c} U^x \]

where

\[ U^x = (q, \gamma \vec{u}) \]

and \( F^{\alpha \beta} = \text{EM field vector} \) = velocity 4-vector.

Alternate methods:

and these reduce to

\[ \frac{d \vec{p}}{d t} = q \left( \vec{E} + \frac{\vec{u}}{c} \times \vec{B} \right) \]

\[ \frac{d E}{d t} = q \vec{u} \cdot \vec{E} \]

where \( \vec{p} = \gamma m_0 \vec{u} \)

\( E = \gamma m_0 c^2 \).

Another way to derive these is using a covariant version of the principle of least action.

Recall from mechanics that the action is

\[ A = \int_{t_1}^{t_2} \left\{ L \right\} d t \]

where \( L = \frac{1}{2} m \frac{d^2 x}{dt^2} \) is the Lagrangian.
\[ \delta A = 0 \]
\[ \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \]

Now, we can rewrite \( A \) as
\[ A = \int_{t_1}^{t_2} \mathcal{L} \left( q_i, \dot{q}_i, t \right) \, dt \]

Now, I claim that \( A \) is a Lorentz invariant \[ \delta A = 0 \text{ along "path of least action"} \]
\[ \Rightarrow \mathcal{L} = \text{Lorentz-invariant} \]
\[ \left( = \text{Lorentz scalar} \right) \]

Case I. Particle of rest mass \( m_0 \),
no potential
\( L \) can depend on \( \vec{u} \) and \( m_0 \) but not position
\[ \mathcal{L} \propto U^a U^a = (\vec{u}, \vec{u}) \cdot (\vec{u}, \vec{u}) \]

Now, \( U^a = (\gamma c, \vec{u}) \)
\[ \text{So } U^a \cdot U^a = \gamma^2 (c^2 - \vec{u}^2) = c^2 \]
Const. of proportionality is \(-m_0\)
\[ \text{So } \mathcal{L} = -m_0 c^2 \]
\[ L = -m_0 c^2 \sqrt{1 - \frac{u^2}{c^2}} \]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial u_x} \right) - \frac{\partial L}{\partial x} = 0 \]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial u_x} \right) = \frac{d}{dt} (\gamma m_0 u_x) = 0 \]
\[ \text{or } \frac{d}{dt} (\gamma m_0 \dot{u}) = 0 \]

So just invariance props. give eq. of motion

What about in an electromagnetic field?

Well, if particle were moving slowly, we would have

\[ L = \frac{1}{2} m_0 u^2 - e \Phi \text{ for charge } e \]

\[ = \frac{1}{2} T - V \text{ since } V = e \Phi \]

We must have a Lagrangian such that \( \delta L \) reduces to this form for small \( \dot{u} \)

Well, the \( T \) part goes to \(-m_0 c^2 \sqrt{1 - \frac{u^2}{c^2}}\)

How about \( \delta L_{\text{int}} \)

\[ \delta L_{\text{int}} \rightarrow -e \Phi \]

\( \Phi \) is part of a 4-vector
Natural generalization is

$$\Delta L_{\text{int}} = -\frac{e}{c} u^\alpha A^\alpha$$

where $A^\alpha = (\Phi, \vec{A})$

and $u^\alpha = (\gamma c, -\gamma \vec{u})$

$$u^\alpha A^\alpha = \gamma \Phi, -\gamma \vec{u} \cdot \vec{A}$$

Hence $\Delta L_{\text{int}} = -e \gamma \Phi + \frac{e}{c} \gamma \vec{u} \cdot \vec{A}$

and $L_{\text{int}} = -e \Phi + \frac{e}{c} \vec{u} \cdot \vec{A}$

So

$$L = -mc^2 \sqrt{1 - \frac{u^2}{c^2}} - e \Phi + \frac{e}{c} \vec{u} \cdot \vec{A}$$

If we write out

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{u}^i} \right) - \frac{\partial L}{\partial u^i} = 0$$

we will get

$$\frac{d}{dt} (\gamma m u^i) = \delta e E^i$$

$$+ \left( \frac{\vec{u}}{c} \times \vec{B} \right)^i$$

Corresponding Hamiltonian is what?

Need canonical momentum

$$p^i = \frac{\partial L}{\partial \dot{u}^i} = \gamma m u^i + \frac{e}{c} A^i$$
or
\[ m_{0} u_{i} = \frac{p_{i}}{m} - \frac{e}{c} A_{i} \]

**NR limit**

\[ u_{i} = \frac{p_{i}}{m} - \frac{e}{m c} (p_{i} - \frac{e}{c} A_{i}) \]

or
\[ \vec{u} = \frac{\vec{p}}{m_{0}} - \frac{e}{m_{0} c} \vec{A} \]

**QM.** \( \hat{p} \rightarrow -i \hbar \hat{\nabla} \) \( \text{Important in QM} \)

---

**In terms of the canonical momentum,**

we have

\[ m_{0} \hat{u} = \hat{p} - \frac{e}{c} \hat{A} \]

\[ \hat{p} = \text{canonical momentum} \]

\[ \hat{p}^{2} m_{0}^{2} u^{2} = (\hat{p} - \frac{e}{c} \hat{A})^{2} \]

\[ = \frac{u^{2}}{1 - u^{2} c^{2}} \]

Solve for \( \hat{u} \):

\[ \frac{\hat{u}}{c} = \frac{\hat{p} - \frac{e \hat{A}}{c}}{\sqrt{(\frac{\hat{p} - e \hat{A}}{c})^{2} + m_{0}^{2} c^{2}}} \]

Finally, **canonical Hamiltonian is**

\[ H = \hat{p} \cdot \hat{u} - L \]
\[ = \frac{(c\vec{p} - e\vec{A})}{\sqrt{(c\vec{p} - e\vec{A})^2 + m^2 c^2}} + mc\sqrt{1 - \frac{u^2}{c^2}} + e\Phi - \frac{e}{c}\vec{u} \cdot \vec{A} \]

Substitute for \( \vec{u} \) in terms of \( \vec{p} \) to get

\[ H = \sqrt{(c\vec{p} - e\vec{A})^2 + m^2 c^4} + e\Phi \]

\( \vec{p} = \text{canonical momentum} \)

NR regime:

\[ \vec{H} \equiv m_0 c^2 \sqrt{1 + \frac{(c\vec{p} - e\vec{A})^2}{m^2 c^4}} + e\Phi \]

\[ \approx m_0 c^2 + \frac{(c\vec{p} - e\vec{A})^2}{2m^2 c^2} + e\Phi \]

\[ = m_0 c^2 + \frac{1}{2m_0} (\vec{p} - \frac{e\vec{A}}{c})^2 + e\Phi \]

\( \vec{p} = \text{canonical momentum} \rightarrow -i\hbar \vec{\nabla} \) in QM.

\[ \vec{p} = \text{canonical momentum} \rightarrow -i\hbar \vec{\nabla} \text{ in QM.} \]

\[ \text{B. Hamilton's eqs. give Lorentz force equations, i.e.} \]

\[ u_i = \frac{\partial H}{\partial p_i} \]

\[ p_i = -\frac{\partial H}{\partial u_i} \]
Some examples of Lorentz force applications.

Motion in static $E$-field starting from $\vec{u} = 0$

$$\frac{d}{dt}(\gamma m_0 \vec{u}) = e\vec{E}$$

So $\gamma m_0 \vec{u} = e\vec{E}t$ \hspace{1cm} Let $\vec{E} = E\hat{\imath}$

$$\vec{u} = \gamma m_0 \vec{u} = eE\hat{\imath} \Rightarrow \vec{u} = u_\gamma \hat{\imath}$$

Then $\vec{u} = \frac{eEt}{m_0}$ \hspace{1cm} already done

$$u(t) = \frac{eEt/m_0}{\sqrt{1 + \left(\frac{eEt}{m_0 c}\right)^2}}$$

Motion in a uniform magnetic field:

Let $\vec{u}$

Eq. of motion is:

$$\frac{d\vec{p}}{dt} = \frac{e}{c} \vec{u} \times \vec{B} \hspace{1cm} \frac{d\vec{E}}{dt} = 0$$

$\Rightarrow \gamma = \text{const.}$

$$\frac{d}{dt}(\gamma m_0 \vec{u}) = \frac{e}{c} \vec{u} \times \vec{B}$$
\[ m \gamma m_0 \frac{d\vec{u}}{dt} = \frac{e}{c} \vec{u} \times \vec{B} \]

\[ \frac{d\vec{u}}{dt} = \frac{e}{\gamma m_0 c} \vec{u} \times \vec{B} \]

Let \( \vec{B} = \vec{B}_z \)

\( u_z = \text{const.} \)

\[ u_x = \frac{eB}{\gamma m_0 c} u_y = w_0 u_y \]

\[ u_y = -\frac{eB}{\gamma m_0 c} u_x = -w_0 u_x \]

\[ u_z = 0 \quad u_z = u_{z0} \]

Try \( \omega_c = \frac{eB}{\gamma m_0 c} = \text{cyclotron freq.} \)

Try \( u_x = u_{x0} e^{-i\omega t} \) where the physical \( u_x \)

is the real part

\[ u_y = u_{y0} e^{-i\omega t} \]

\[-i \omega u_{x0} = \omega u_{y0} \]

\[-i \omega u_{y0} = -\omega u_{x0} \]

\[ u_x = i \omega u_{y0} = i u_{y0} e^{-i\omega t} \]

\[ u_y = u_{y0} e^{-i\omega t} \]

\[ x = x_0 + u_{y0} e^{-i\omega t} \frac{z - z_0}{\omega_c} \]

\[ y = y_0 + \frac{i u_{y0}}{\omega_c} e^{-i\omega t} \]
More details:

In lab. frame $K$:

\[
\frac{dp}{dt} = e \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)
\]

In primed frame $K'$:

\[
\frac{dp'}{dt'} = e \left( \vec{E}' + \frac{\vec{v}}{c} \times \vec{B}' \right)
\]

Say $\vec{E} \perp \vec{B}$ and $|\vec{E}| < |\vec{B}|$

Choose $\vec{v} = \text{velocity of } K'$ compared to $K$

If we choose $\vec{v} = c \frac{\vec{E} \times \vec{B}}{B^2}$ then $\vec{E}' = 0$

In moving frame only have a $B$-field

Back in lab, we have a spiral spiral motion

Superimposed on a drift in the $\vec{E} \times \vec{B}$ direction

Called $\vec{E} \times \vec{B}$ drift
If $|E| > |B|$ 

Choose \( \mathbf{v} = c \frac{E \times \mathbf{B}}{|E|^2} \)

Eliminate $\mathbf{B}$-field

In this case, we will have an increasing velocity $\mathbf{v}$ of charge in hyperbolic motion in a minimum frame.
\( |E| \leq |B| \); parametric eqs. of motion

Let \( E = E_x \)
\( B = B_y \)

Then the velocity which eliminates \( E \) is

\[
\vec{v} = \frac{c E_x \vec{B}}{|B|^2} = \frac{c E_x}{B} \vec{z} = v \vec{z}
\]

In primed frame \( E' = 0 \)
\( B' = \sqrt{B^2 - E^2} \)

Orbit in primed frame is (taking \( x_0, y_0, z_0 = 0 \))

\[
x'(t') = -\frac{u_1}{\omega_c} \frac{ut}{\omega_c} \cos \omega_c t'
\]

\[
y' = \frac{u_1}{\omega_c} \frac{ut}{\omega_c} \sin \omega_c t'
\]

\[
z' = \frac{u_0}{\omega_c} t'
\]

\( \omega_c = \frac{EB'}{\omega_c m_0 c} \)

Now transform back into lab

\[
x = x'(t') = -r_0 \cos \omega_c t'
\]

\[
y = y' = r_0 \sin \omega_c t'
\]

\[
z = \sigma u (z' + vt')
\]

\[
= \sigma u (u_0 + v) t'
\]

where \( \sigma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \)

\[
\sigma_0 = \frac{1}{\sqrt{1 - \frac{E^2}{B^2}}}
\]

This gives orbit in parametric form, with \( t' \) being the parameter.
Lagrangian for Field:

First, we have that
\[ L = \int L \, d^3x \]

where \( L \) = Lagrangian density
(for field, given currents and charge densities)

Then the action is
\[ A = \int_{t_1}^{t_2} \int L \, d^3x \, dt \]

Now, we know that \( A \) should be a Lorentz-invariant.
We also know that the volume element
\[ dt \, d^3x = ct \, d^3x \, dt \]
transforms as follows under a Lorentz transformation

\[ d^4x' = \sqrt{\gamma} \, dx_0' \, dx_1' \, dx_2' \, dx_3' \]

But, \( dx_0' \, dx_1' \, dx_2' \, dx_3' = \sqrt{\gamma} \, \gamma^{01} \, \gamma^{02} \, \gamma^{03} \, dx_0 \, dx_1 \, dx_2 \, dx_3 \)

where
\[
\frac{\partial (x_0', x_1', x_2', x_3')}{\partial (x_0, x_1, x_2, x_3)} = \begin{vmatrix}
\frac{\partial x_0'}{\partial x_0} & \frac{\partial x_0'}{\partial x_1} & \frac{\partial x_0'}{\partial x_2} & \frac{\partial x_0'}{\partial x_3} \\
\frac{\partial x_1'}{\partial x_0} & \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} & \frac{\partial x_1'}{\partial x_3} \\
\frac{\partial x_2'}{\partial x_0} & \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} & \frac{\partial x_2'}{\partial x_3} \\
\frac{\partial x_3'}{\partial x_0} & \frac{\partial x_3'}{\partial x_1} & \frac{\partial x_3'}{\partial x_2} & \frac{\partial x_3'}{\partial x_3}
\end{vmatrix}
\]
\[ \text{det} \left[ A \right] \text{ where } A \text{ is the matrix for the Lorentz transformation} \]

Since we are interested in a proper Lorentz transformation, \( d^4x \) is a Lorentz invariant and \( d^4x' = d^4x \).

Finally, \( L \) is a Lorentz-invariant.

How do we construct a Lorentz-invariant out of the field tensor \( F \)?

Well, there is \( F_{\mu\nu} F^{\mu\nu} \propto E^2 - B^2 \)

There is also \( J_{\mu\nu} F^{\mu\nu} \propto E \cdot B \)

But this term doesn't enter the Lagrangian density (not invariant under the coordinate transformation)

\[
\begin{align*}
\xi &\to -\xi \\
\eta &\to -\eta \\
\tau &\to -\tau
\end{align*}
\]

If there is also a current density, we must add a term related to \( J_\mu \), which is Lorentz-invariant. This will be proportional to \( J_\mu A_\mu \).
Correct Lagrangian density turns out to be

\[ L = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_{\mu} A^{\mu} \]

How to get eqs. of motion (i.e. Maxwell's eqs.) for these equations.

To see how to produce these eqs., consider Lagrangian density for a string

\[ L = \frac{\rho}{2} (\frac{\partial u}{\partial t})^2 - \frac{K}{2} (\frac{\partial u}{\partial x})^2 = K - V \]

\[ K = K.E. \text{ density/length} \]

\[ V = P.d. \text{ energy density/length} \]

For a single degree of freedom, the Lagrange eqs. of motion are

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial q_i'} \right) - \frac{\partial L}{\partial q_i} = 0 \]

For any many degrees of freedom, corresponding eq. is

\[ \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial u_t} \right) + \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial u_x} \right) - \frac{\partial L}{\partial u} = 0 \]

[See Goldstein, ch. 12]
\[ \sum_{\nu=0}^{3} \frac{\partial}{\partial x^{\nu}} \frac{\partial L}{\partial \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \right)} + \frac{\partial L}{\partial A^{\mu}} = 0 \]

\[ \frac{\partial L}{\partial A^{\mu}} = -\frac{J^{\mu}}{c} \]

Now the free-field part of \( L \) is

\[ -\frac{1}{16\pi} \left[ \epsilon_{\mu
u} A^{\nu} - \partial_{\nu} A_{\mu} \right] \left[ \epsilon^{\mu\nu} A^{\nu} - \partial^{\nu} A_{\mu} \right] \]

and when one calculates

\[ \sum_{\nu=0}^{3} \frac{\partial}{\partial x^{\nu}} \frac{\partial L}{\partial \left( \frac{\partial A^{\mu}}{\partial x^{\nu}} \right)} \] one finds that it is just

\[ \frac{1}{4\pi} \frac{\partial}{\partial x_{\nu}} F_{\mu \nu} \]

As a result, one obtains for the four equations of motion

\[ \frac{1}{4\pi} \frac{\partial}{\partial x_{\nu}} F_{\mu \nu} = \frac{1}{c} J_{\mu} \]

which are the 4 inhomogeneous Maxwell equations.
Example of how to get eq. of motion from Lagrangian density:

\[ L = \frac{1}{8\pi} \left( E^2 - B^2 \right) + \frac{1}{\epsilon} \vec{E} \cdot \vec{B} - \rho \Phi \]

One of the equations is

\[ \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial (\partial \Phi / \partial x)} \right) + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( \frac{\partial L}{\partial (\partial \hat{\Phi} / \partial x_i)} \right) + \frac{\partial L}{\partial \Phi} = 0 \]

\[ \frac{\partial L}{\partial \Phi} = -\rho \]

\[ \frac{\partial \Phi}{\partial (\partial \Phi / \partial x)} = 0. \]

\[ \frac{\partial L}{\partial (\partial \hat{\Phi} / \partial x_i)} = \frac{\partial \Phi}{\partial (\partial \hat{\Phi} / \partial x_i)} \left[ \frac{1}{8\pi} \left( -\hat{\nabla} \Phi - \frac{1}{\epsilon} \frac{\partial A}{\partial t} \right) \cdot \left( \hat{\nabla} \Phi \right) \right] \]

\[ = -\frac{1}{4\pi} \left( -\frac{\partial \Phi}{\partial x_i} - \frac{1}{\epsilon} \frac{\partial A}{\partial t} \right) = -\frac{E_i}{4\pi} \]

\[ -\sum_i \frac{\partial E_i}{\partial x_i} = -\hat{\nabla} \cdot \vec{E} \quad \text{so} \]

\[ \frac{\nabla \cdot \vec{E}}{4\pi} - \rho = 0 \]

\[ \nabla \cdot \vec{E} = -4\pi \rho \]