Radiation By Harmonically Oscillating Sources

Showed earlier this quarter that the retarded solutions to the equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi(\vec{x}\,t) = -\frac{\rho(x,t)}{\epsilon_0}$$

is

$$\Phi(\vec{x}\,t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \int dt' \frac{\rho(x'\,t')}{|\vec{x}-\vec{x}'|}\,\delta\left(t' - t + \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

and that the retarded solution to

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{x}t) = -\mu_0\vec{J}(\vec{x}t)$$

is

$$\vec{A}(\vec{x}t) = \frac{\mu_0}{4\pi}\int d^3x' \int dt' \frac{\vec{J}(\vec{x}'\,t')}{|\vec{x}-\vec{x}'|}\,\delta\left(t' - t + \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

Given $\vec{A}$ and Φ, can get $\vec{B}$ and $\vec{E}$.

Now suppose we have

$$\rho(\vec{x}t) = \rho(\vec{x})e^{-i\omega t}$$
$$\vec{J}(\vec{x}t) = \vec{J}(\vec{x})e^{-i\omega t}$$

We will calculate fields in this case.

First, we need

$$\vec{A}(\vec{x}t) = \frac{\mu_0}{4\pi}\int d^3x'\,\vec{J}(\vec{x}')\frac{1}{|\vec{x}-\vec{x}'|}\int_{-\infty}^{\infty} dt'\,e^{-i\omega t'}\,\delta\left(t' - t + \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

The time integral is

$$e^{-i\omega\left(t \mp \frac{|\vec{x}-\vec{x}'|}{c}\right)}$$

so

$$\vec{A}(\vec{x}t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} e^{+ik|\vec{x}-\vec{x}'|}$$

where $k = \frac{\omega}{c}$

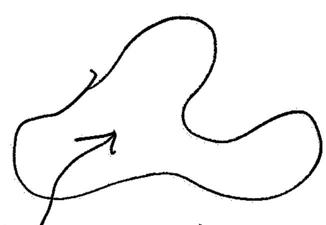
$$= \vec{A}(\vec{x}) e^{-i\omega t}$$

where

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} e^{ik|\vec{x}-\vec{x}'|}$$

with similar eq. for Φ (which we won't need)

Localized source:



$\vec{J}(\vec{x}')$, $\rho(\vec{x}')$

In general, $\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}$

and $\vec{E}$ can be gotten from

$\cancel{\frac{\partial \vec{H}}{\partial t}}$

$$\vec{\nabla} \times \vec{H} = + \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$= \frac{\partial \vec{D}}{\partial t} \quad \text{outside the source}$$

$$= -i\omega\epsilon_0 \vec{E}$$

So $\vec{E}(\vec{x}) = \frac{i}{\omega\epsilon_0} \vec{\nabla} \times \vec{H}(\vec{x})$ outside the source.

For localized source, let $d \sim$ size of source, and define the following three zones, if $d \ll \lambda = \frac{2\pi}{k}$

(i). Near zone: $d \ll r \ll \lambda$

(ii). Intermediate zone: $d \ll r \sim \lambda$

(iii) Far (radiation) zone

$d \ll \lambda \ll r$.

Near zone: $k|\vec{x}-\vec{x}'| \ll 1$ so

$$\vec{A}(\vec{x}) \sim \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

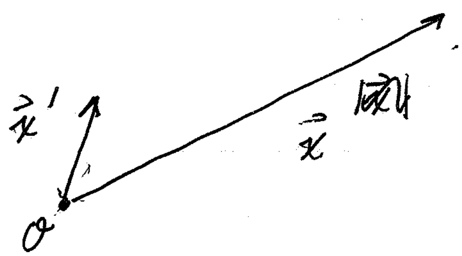
Field looks like a static field, as in chapter 5, except for an $e^{-i\omega t}$ time dependence.

For fields:

$$\frac{1}{|\vec{x}-\vec{x}'|} \sim \frac{1}{|\vec{x}|} = \frac{1}{r}$$

and

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi r}\int \vec{J}(\vec{x}')\, e^{+ik|\vec{x}-\vec{x}'|}\, d^3x'$$



$$k\,|\vec{x}-\vec{x}'| = k\left[\,|\vec{x}|^2 - 2\vec{x}\cdot\vec{x}' + |\vec{x}'|^2\right]^{1/2}$$

$$= k\,|\vec{x}|\left[1 - \frac{2\vec{x}\cdot\vec{x}'}{|\vec{x}|^2} + \frac{|\vec{x}'|^2}{|\vec{x}|^2}\right]^{1/2} \quad \text{still exact}$$

$$\approx k|\vec{x}|\left[1 - \frac{1}{2}\left(\frac{2\vec{x}\cdot\vec{x}'}{|\vec{x}|^2} - \frac{|\vec{x}'|^2}{|\vec{x}|^2}\right) + \mathcal{O}\left(\left(\frac{|\vec{x}'|}{|\vec{x}|^2}\right)^2\right)\right] \quad \text{approximate}$$

$$\approx \underbrace{k|\vec{x}|}_{=kr} - \frac{k\,\vec{x}\cdot\vec{x}'}{|\vec{x}|} + \cancel{\frac{k|\vec{x}'|^2}{|\vec{x}|}} + \mathcal{O}\ldots$$

neglect

$$= kr - k\hat{n}\cdot\vec{x}' \quad \text{where } \hat{n} = \frac{\vec{x}}{|\vec{x}|} = \frac{\vec{x}}{r}$$

Thus

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi r}\, e^{ikr}\int d^3x'\, e^{-ik\hat{n}\cdot\vec{x}'} \qquad (1)$$

general result for far zone

Expand exponential in power series
(OK for small source)

Actually, eq. (1) is valid even for large source so long as $r \gg \lambda$ and $r \gg d$ even if d is not $\ll \lambda$.

But for small source, the expansion is useful:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_{m=0}^{\infty} \frac{(-ik)^m}{m!} \int d^3x' \left(\hat{n}\cdot\vec{x}'\right)^m \vec{J}(\vec{x}')$$

Now what? First assume small source and just keep $m=0$ term

Need $\int \vec{J}(\vec{x}')\, d^3x'$

Do an integration by parts. Consider

$$\int_{\text{source}} J_i(\vec{x}')\, d^3x' = \int_{\text{source}} \vec{J}\cdot\nabla' x_i\, d^3x' \quad \text{since} \quad \vec{\nabla}' x_i = \hat{x}_i$$

$$= \underbrace{\int \vec{\nabla}'\cdot(\vec{J}x_i)\, d^3x'}_{= \oint_S J_n x_i\, d^3x} - \int x_i' \vec{\nabla}'\cdot\vec{J}\, d^3x'$$

where S is outside source
$= 0$

But the continuity eq. is

$$-\frac{\partial \rho}{\partial t} = \vec{\nabla}' \cdot \vec{J} = +i\omega\rho$$

So finally

$$\int J_i(\vec{x}')\, d^3x' = -i\omega \int x_i' \rho(\vec{x}')\, d^3x'$$

and hence

$$\int \vec{J}(\vec{x}')\, d^3x' = -i\omega \int \vec{x}' \rho(\vec{x}')\, d^3x' = -i\omega \vec{p}$$

So $$\vec{A}(\vec{x}) = -\frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} i\omega \vec{p}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A} = -\frac{\cancel{\mu_0}\,1}{4\pi} i\omega \vec{\nabla}\left(\frac{e^{ikr}}{r}\right) \times \vec{p}$$

$$= -\frac{\cancel{\mu_0}\,1}{4\pi} i\omega \left[\frac{ik e^{ikr}}{r} - \underbrace{\cancel{\frac{e^{ikr}}{r^2}}}_{\text{neglect}}\right] \hat{n} \times \vec{p}$$

So $$\vec{H} = +\frac{ck^2}{4\pi r} e^{ikr}\, \hat{n} \times \vec{p}$$ using $\omega = ck$

Then $$\vec{E} = \frac{i}{\omega \epsilon_0} \vec{\nabla} \times \vec{H}$$

$$\simeq \frac{i}{\omega \epsilon_0}(+ik\,\hat{n} \times \vec{H}) = \frac{k}{\omega \epsilon_0} \vec{H} \times \hat{n}$$

But $$\frac{k}{\omega \epsilon_0} = \frac{1}{c\epsilon_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} = Z_0 =$$ "impedance of free space"

Finally, the radiated power per unit <u>area</u> (time-averaged) is

$$\vec{S}_{av}\cdot\hat{n} = \frac{1}{2}\,\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right)\cdot\hat{n}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{Z_0\left(\vec{H}\times\hat{n}\right)\times\vec{H}^*\right\}\cdot\hat{n}$$

$$= \frac{Z_0}{2}\,|\vec{H}|^2$$

$$= \frac{Z_0}{2}\,\frac{c^2k^4}{(4\pi)^2 r^2}\,|\hat{n}\times\vec{p}|^2$$

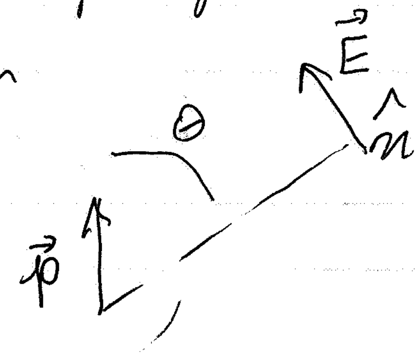
$\frac{dP_{av}}{d\Omega}$ = power <u>radiated</u> <u>per</u> <u>unit</u> <u>solid</u> <u>angle</u>

$$= \vec{S}_{av}\cdot\hat{n}\,r^2$$

$$= \frac{k^4c^2Z_0}{32\pi^2}\left|\left(\hat{n}\times\vec{p}\right)\times\hat{n}\right|^2$$

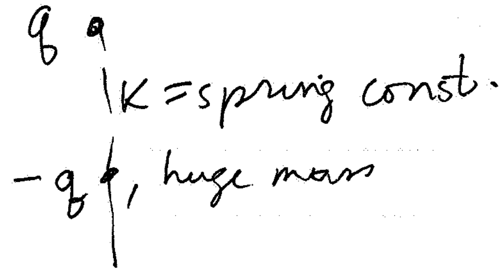
where we write $|\hat{n}\times\vec{p}|^2 = |(\hat{n}\times\vec{p})\times\hat{n}|^2$ to indicate direction of polarization of $\vec{E}$

Typical dipole pattern. Suppose $\vec{p}$ along a line.
Then



$$\frac{dP_{av}}{d\Omega} = \frac{c^2 k^4 Z_0}{32\pi^2} |\vec{p}|^2 \sin^2\theta$$

E. g. oscillating charge



$$\vec{p} = p_0 \hat{z} e^{-i\omega t} \qquad \omega = \sqrt{\frac{k}{m}}$$

Integrated radiated power involves

$$\int_0^{\pi} \int_0^{2\pi} \sin^2\theta \sin\theta \, d\theta \, d\phi$$

$$= 2\pi \int_{-1}^{1} (1-x^2)\, dx = 2\pi \left[x - \frac{x^3}{3} \right]_{-1}^{1}$$

$$= \frac{8\pi}{3}$$

So $$P_{av} = \frac{c^2 k^4 Z_0}{12\pi^2} |\vec{p}_0|^2$$

$$= \frac{c^2 k^4 Z_0}{12\pi^2} (q a_0)^2$$

Compare with ~~potential~~ max potential energy: $\frac{1}{2} m\omega^2 a_0^2 = \frac{1}{2} mc^2 k^2 a_0^2$

$$\frac{P_{av}}{U} = \frac{c^2 k^4 Z_0}{12\pi^2} (q a_0)^2 \Big/ \left[\frac{1}{2} mc^2 k^2 a_0^2\right]$$

$$= \frac{Z_0}{24\pi^2} \frac{k^2 q^2}{m}$$

At short distances,

$$\Phi(\vec{x}) \sim \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

So just looks like static electric dipole field (but with $e^{-i\omega t}$ time-dependence)

MT High 96(2)
Median 68
Low 26

Magnetic dipole, electric quadrupole radiation

Second term in expansion is

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')(1 - ik\hat{n}\cdot\vec{x}')d^3x'$$

Now use

$$\vec{J}(\vec{x}')\hat{n}\cdot\vec{x}' = \frac{1}{2}\left[\hat{n}\cdot\vec{x}'\,\vec{J}(\vec{x}') + \frac{1}{2}(\hat{n}\cdot\vec{J})\vec{x}'\right] + \frac{1}{2}\underbrace{\left[\hat{n}\cdot\vec{x}'\,\vec{J}(\vec{x}') - \frac{1}{2}(\hat{n}\cdot\vec{J})\vec{x}'\right]}_{(x'\times\vec{J})\times\hat{n}}$$

First consider the second term. This is

$$\vec{A}_{MD} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left[\frac{1}{2}\int \vec{x}'\times\vec{J}(\vec{x}')\,d^3x'\times\hat{n}\right]$$

$$= -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(\vec{m}\times\hat{n}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{m}$$

$$\vec{m} = \frac{1}{2}\int \vec{x}'\times\vec{J}(\vec{x}')\,d^3x'$$

$$\vec{H}_{MD} = \vec{\nabla}\times\vec{A}_{MD}$$

$$= \frac{k^2}{4\pi}(\hat{n}\times\vec{m})\times\hat{n}\,\frac{e^{ikr}}{r} + O\left(\frac{1}{r^2}\right)$$

Get $\vec{E}_{MD}$ from $\qquad \frac{\partial \vec{D}}{\partial t} = \vec{\nabla} \times \vec{H}$

$$= -i\omega \epsilon_0 \vec{E}$$

So $\vec{E} = \frac{i}{\omega \epsilon_0} \nabla \times \vec{H}_{MD}$

$$= -\frac{Z_0}{4\pi} k^2 (\hat{n} \times \vec{m}) \frac{e^{ikr}}{r} + \theta\left(\frac{1}{r^2}\right)$$

where $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$

using the identity

$\vec{\nabla} \times (f \vec{V}) = \vec{\nabla} f \times \vec{V}$ for a const. vector $\vec{V}$

or here

$$\vec{E} = \frac{i}{\omega \epsilon_0} \frac{k^2}{4\pi} \vec{\nabla}\left(\frac{e^{ikr}}{r}\right) \times [(\hat{n} \times \vec{m}) \times \hat{n}]$$

$$= -\frac{k^3}{\omega \epsilon_0} \frac{1}{4\pi} \frac{e^{ikr}}{r} \hat{n} \times [(\hat{n} \times \vec{m}) \times \hat{n}]$$

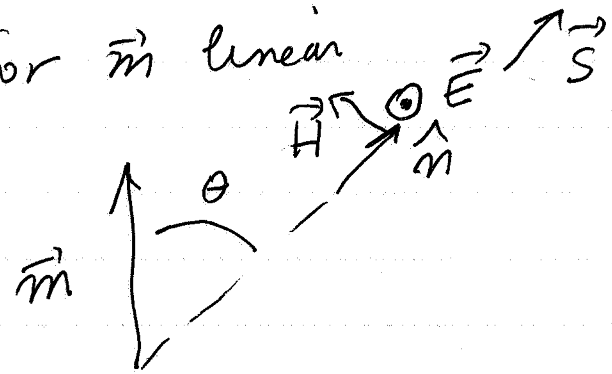
$$= \frac{Z_0}{4\pi} k^2 \hat{n} \times \vec{m} \frac{e^{ikr}}{r}$$

using $\omega = ck = \frac{1}{\sqrt{\mu_0 \epsilon_0}} k$

Then $\frac{dP_{av}}{d\Omega} = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)\cdot\hat{n}\, r^2$

$$= \frac{1}{2}\frac{Z_0}{(4\pi)^2}k^4|\hat{n}\times\vec{m}|^2 = \frac{Z_0}{32\pi^2}k^4|\hat{n}\times\vec{m}|^2$$

E.g. for $\vec{m}$ linear



$$\frac{dP_{av}}{d\Omega} = \frac{Z_0}{32\pi^2}k^4|\vec{m}|^2\sin^2\theta$$

Symmetric term gives electric quadrupole contribution

First rearrange the term

$$\frac{1}{2}\int\left[(\hat{n}\cdot\vec{x}')J_i + (\hat{n}\cdot\vec{J})x_i'\right]d^3x'$$ as follows:

$$\cancel{= \frac{1}{2}\sum_{j=1}^{3}\int\left[J_i x_j' + J_j x_i'\right]n_j\, d^3x'}$$

$$\cancel{= \frac{1}{2}\sum_{j=1}^{3}\int(J_i x_j' + J_i x_j')\,\hat{x}_j\cdot\hat{n}\, d^3x'}$$

$$\frac{1}{2}\int\left[(\hat{n}\cdot\vec{x}')J_i + (\hat{n}\cdot\vec{J})x_i'\right]d^3x'$$

$$= \frac{1}{2}\int \underbrace{\vec{\nabla}'\left(x_i'(\hat{n}\cdot\vec{x}')\right)}_{= x_i'\hat{n} + (\hat{n}\cdot\vec{x}')\hat{x}_i}\cdot\vec{J}(\vec{x}')\,d^3x'$$

$$= \frac{1}{2}\int\vec{\nabla}'\cdot\left(x_i'(\hat{n}\cdot\vec{x}')\vec{J}(\vec{x}')\right)d^3x'$$

$$0 \qquad - \frac{1}{2}\int x_i'(\hat{n}\cdot\vec{x}')\underbrace{\vec{\nabla}'\cdot\vec{J}}_{= i\omega\rho(\vec{x}')}\,d^3x'$$

So $\frac{1}{2}\int\left[(\hat{n}\cdot\vec{x}')\vec{J}(\vec{x}') + \frac{1}{2}(\hat{n}\cdot\vec{J})\vec{x}'\right]$

$$= -\frac{1}{2}\int\vec{x}'(\hat{n}\cdot\vec{x}')\rho(\vec{x}')\,d^3x$$

and the contribution of this term to $\vec{A}$ is

$$\vec{A}_{EQ} = -\frac{\mu_0 ck^2}{8\pi}\frac{e^{ikr}}{r}\int\vec{x}'(\hat{n}\cdot\vec{x}')\rho(\vec{x}')\,d^3x' + \theta\left(\frac{1}{r^2}\right)$$

Then $\vec{H}_{EQ} = \vec{\nabla}\times\vec{A}_{EQ}/\mu_0$

$$= \frac{ik\hat{n}\times\vec{A}_{EQ}}{\mu_0}$$

$$\vec{E}_{EQ} = \frac{k}{\omega\epsilon_0}\hat{n}\times\vec{H}_{EQ} = \frac{ikZ_0(\hat{n}\times\vec{A}_{EQ})\times\hat{n}}{\mu_0}$$

Can rewrite in terms of electric quadrupole tensor

$$Q_{\alpha\beta} = (3x'_\alpha x'_\beta - r'^2 \delta_{\alpha\beta})\rho(\vec{x}')\, d^3x'$$

Eventually get

$$\vec{H}_{EQ} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{Q}(\hat{n})$$

where $Q_\alpha(\hat{n}) = \sum_\beta Q_{\alpha\beta} n_\beta$

$$\frac{dP_{av}}{d\Omega} = \frac{c^2 Z_0 k^6}{\underbrace{1152}\pi^2}\left|\hat{n}\times\vec{Q}(\hat{n})\right|^2$$

$24^2\cdot 2 = 576\cdot 2$

$\vec{E}_{EQ} \parallel (\hat{n}\times\vec{Q})\times\hat{n}$

$$P_{av} = \int \frac{dP_{av}}{d\Omega}\, d\Omega$$

= (see Jackson)

$$= \frac{c^2 Z_0 k^6}{1440\pi}\sum_{\alpha\beta}|Q_{\alpha\beta}|^2$$