Model For Optical Rotation Due to A Magnetic Field

$$m\ddot{\vec{x}} = \vec{F} = -e\vec{E} - e\dot{\vec{x}} \times \vec{B} - m\omega_0^2\vec{x}$$

$$m\left[\ddot{\vec{x}} + \frac{e}{m}\dot{\vec{x}} \times \vec{B} + \omega_0^2\vec{x}\right] = -e\vec{E}e^{-i\omega t}$$

Let $\vec{x} = \vec{x}_0 e^{-i\omega t}$

$$m\left[(-\omega^2+\omega_0^2)\vec{x}_0 - i\omega\frac{\vec{x}_0 \times \vec{B}e}{m}\right] = -e\vec{E}$$

Take $\vec{B} = B\hat{z}$

In component form,

$$m\left[(-\omega^2+\omega_0^2)x_0 - \frac{ie\omega y_0 B}{m}\right] = -eE_x$$

$$m\left[(-\omega^2+\omega_0^2)y_0 + \frac{ie\omega x_0 B}{m}\right] = -eE_y$$

or

$$\begin{bmatrix} -\omega^2+\omega_0^2 & -i\omega\omega_c \\ +i\omega\omega_c & -\omega^2+\omega_0^2 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = -\frac{e}{m}\begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

Solve for x_0 and y_0:

$$x_0 = \left[(\omega_0^2-\omega^2)^2 - \omega^2\omega_c^2\right]^{-1}\left[E_x(\omega_0^2-\omega^2) + i\omega\omega_c E_y\right]\left(-\frac{e}{m}\right)$$

$$y_0 = \left[(\omega_0^2-\omega^2)^2 - \omega^2\omega_c^2\right]^{-1}\left[-i\omega\omega_c E_x + (\omega_0^2-\omega^2)E_y\right]\left(-\frac{e}{m}\right)$$

Case I: $E_y = +iE_x$

Then $$x_0 = \left[(\omega_0^2-\omega^2)^2 - \omega^2\omega_c^2\right]^{-1}(\omega_0^2 - \omega^2 - \omega\omega_c)\left(-\frac{e}{m}\right)E_x$$

$$= (\omega_0^2 - \omega^2 + \omega\omega_c)^{-1}\left(-\frac{e}{m}\right)E_x$$

$$y_0 = i x_0$$

$$P_x = \frac{ne^2/m}{\omega_0^2-\omega^2+\omega\omega_c} E_x$$

$$P_y = i P_x$$

$$P_+ = P_x + iP_y = \frac{ne^2/m}{\omega_0^2-\omega^2+\omega\omega_c}(E_x + iE_y)$$

$$D_+ = \epsilon_0(E_x + iE_y) + (P_x + iP_y)$$

$$= \epsilon_0\left[1 + \frac{ne^2/(m\epsilon_0)}{\omega_0^2-\omega^2+\omega\omega_c}\right]E_+ = \epsilon_0\epsilon_+ E_+$$

Similarly, if $E_y = -iE_x$ we find

$$\epsilon_- = 1 + \frac{ne^2/(m\epsilon_0)}{\omega_0^2-\omega^2-\omega\omega_c}$$

So there is a difference in polarization between + and − wave propagation

For present case

$$n_+ = \sqrt{\epsilon_+}$$

$$n_+ - n_- = \sqrt{\epsilon_+} - \sqrt{\epsilon_-}$$

$$\epsilon_+ \sim 1 + \frac{ne^2/m\epsilon_0}{\omega_0^2 - \omega^2} + \frac{(ne^2/m\epsilon_0)\,\omega\omega_c}{(\omega_0^2-\omega^2)^2}$$

$$\epsilon\ \cancel{n_+ \approx} = n_{\cancel{0}}^2 + \Delta = n^2\left(1 + \frac{\Delta}{n^2}\right)$$

$$\epsilon_- = n_{\cancel{0}}^2 - \Delta$$

$$n_+ \cong n\left(1 + \frac{\Delta}{2n^2}\right)$$

$$n_- \approx n\left(1 - \frac{\Delta}{2n^2}\right)$$

$$n_+ - n_- \sim \frac{\Delta}{n} = \frac{(ne^2/m\epsilon_0)\,\omega\omega_c\,/(\omega_0^2-\omega^2)^2}{\sqrt{1 + (ne^2/m\epsilon_0)/(\omega_0^2-\omega^2)}}$$

$$\propto B \propto \omega_c$$

So angle of rotation is proportional to B.

Optical rotation

Let's say we have a linearly polarized wave

$$E = E_0 \hat{x} \quad \text{at position } z = 0$$

We write $E_0 \hat{x} = \frac{1}{2} E_0 (\hat{x} + i\hat{y})$

$$+ \frac{1}{2} E_0 (\hat{x} - i\hat{y})$$

Index of refraction for + and − polarized waves is assumed to be n_+ and n_-.

Therefore

$$\vec{E}(\vec{x}) = \frac{1}{2} E_0 (\hat{x} + i\hat{y})\, e^{(ik_+ z - i\omega t)}$$

$$+ \frac{1}{2} E_0 (\hat{x} - i\hat{y})\, e^{i(k_- z - i\omega t)}$$

where $k_\pm = n_\pm \frac{\omega}{c}$ and $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$

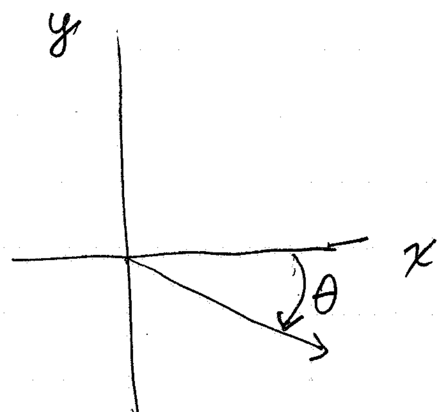
At position z, total wave is

$$\left[\frac{1}{2} E_0 e^{ik_+ z} (\hat{x} + i\hat{y}) + \frac{1}{2} E_0 e^{+ik_- z} (\hat{x} - i\hat{y})\right] e^{-i\omega}$$

$$= \frac{E_0}{2} \left(e^{ik_+ z} + e^{+ik_- z}\right) \hat{x} + \frac{iE_0}{2} \left(e^{ik_+ z} - e^{ik_- z}\right) \hat{y}\, e^{-i\omega t}$$

$$= \frac{E_0}{2} e^{i(k_+ + k_-)z/2 - i\omega t} \left\{ \hat{x} \left(e^{i(k_+ - k_-)z/2} + e^{-i(k_+ - k_-)z/2} \right) + i\hat{y} \left(e^{i(k_+ - k_-)z/2} - e^{-i(k_+ - k_-)z/2} \right) \right\}$$

$$= E_0 e^{i(k_+ + k_-)z/2 - i\omega t} \left\{ \hat{x} \cos \frac{(k_+ - k_-)z}{2} - \hat{y} \sin \frac{(k_+ - k_-)z}{2} \right\}$$



$\theta = \frac{(k_+ - k_-)z}{2}$ ~~so rotates~~

$$\boxed{\frac{d\theta}{dz} = \frac{k_+ - k_-}{2}}$$

~~For~~