Physics 834

Review Topics

For Midterm

Basic equations

$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}\Phi$$

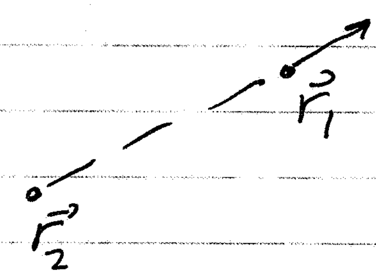
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \Rightarrow \quad \nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

(Poisson's eq.)

Coulomb's Law

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}^3} \vec{r}_{12}$$ = force on charge (1) due to (2)

where $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$



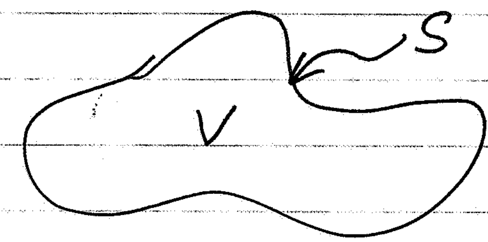
If charge density is known in all space,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

In finite region of space, we have boundary conditions

Dirichlet problem: $\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$ in V

Φ specified on S



Potential Φ is determined uniquely

Neumann problem

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \quad \text{in } V$$

$\frac{\partial \Phi}{\partial n}$ specified on S.

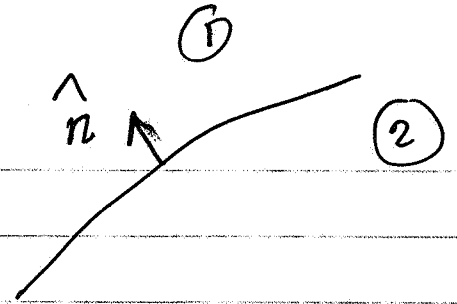
Φ is determined to within a const.

Conductors

A conductor has free charge.

Φ = const. on a conductor (so $\vec{E} = 0$)

charge is confined to outer surface of a conductor



At boundary between two conductors,

$$\vec E_{1t} = \vec E_{2t}$$

and $(\vec E_1 - \vec E_2)\cdot\hat n = \frac{\sigma}{\epsilon_0}$ where

σ = surface charge density

Special case: medium 2 = conductor

Then $\vec E_2 = 0 \Rightarrow E_{1n} = \frac{\sigma}{\epsilon_0}$; $\vec E_{1t} = 0$.

Force on surface of conductor — $\vec F = \frac{1}{2}\sigma \vec E$ per unit area $= \frac{\sigma^2}{2\epsilon_0}$ normally outward

Boundary value problems in electrostatics: how to solve them?

Say want to solve Dirichlet problem in vol V surrounded by surface S.

Some methods include:

1. Green's function method (I haven't used this much)

$$\Phi(\vec x) = \frac{-1}{4\pi}\int_S \Phi(\vec x')\frac{\partial G_D}{\partial n'}d^2x' + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec x')\,G_D(\vec x,\vec x')\,d^3x'$$

where $\nabla'^2 G_D(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$ $\vec{x}'$ in V

$G_D(\vec{x},\vec{x}') = 0$ $\vec{x}'$ on S.

2. Method of images: find a set of images which satisfy boundary conditions

3. (for Laplace's equation) Expand in a suitable complete set of functions ~~for~~ which satisfy Laplace's eq in the relevant coord system.

Energy in electrostatics:

$$W = \frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x$$

$$= \frac{\epsilon_0}{2}\int |\vec{E}(\vec{x})|^2\,d^3x$$

Capacitance. If we have n conductors at voltage $V_1 \ldots V_n$

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j \qquad C_{ij} = \text{capacitance matrix}$$

$$W = \frac{1}{2}\sum_i Q_i V_i = \frac{1}{2}\sum_i\sum_j C_{ij} V_i V_j$$

For two conductors at ~~[illegible]~~ p.
with charge Q, $-Q$,

$$W = \frac{1}{2} C V^2$$ where V = voltage difference

Method of images:

Can often satisfy boundary conditions by images.
(several examples in class).

Separation of variables

Rectangular coordinates

$$\nabla^2 \Phi = \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0$$

General solution is

$$\Phi(x, y, z) = e^{\pm i k_1 x} e^{\pm i k_2 y} e^{\pm k_z z}$$

where $k_3 = \sqrt{k_1^2 + k_2^2}$

Values of k_1, k_2, k_3 determined by boundary conditions

2D : Rectangular coordinates

$$\Phi(x,y) = e^{\pm i k_1 x} e^{\pm k_1 y}$$

if $\nabla^2 \Phi = 0$

2D Polar coordinates

$$\Phi(\rho,\phi) = R(\rho)\,\psi(\phi)$$

$R(\rho) = a\rho^{\nu} + b\rho^{-\nu}$ $\quad (\nu \neq 0)$

$= a + b\ln\rho \quad \nu = 0$

$\psi = A\cos\nu\phi + B\sin\nu\phi$

$\nu =$ integer if all ϕ allowed

$= A + B\phi \quad \nu = 0$

So a typical sol$^{\underline{n}}$ is

$$\left(a\rho^{\nu} + b\rho^{-\nu}\right)\left(A\cos\nu\phi + B\sin\nu\phi\right)$$

3D spherical coords, azimuthal symmetry

If all angles allowed,

$$\psi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

~~One more example:~~

Complete sets of functions

The $P_\ell(x)$'s are complete and orthogonal on $-1 \leq x \leq 1$

i.e. $\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\,dx = 0 \quad \ell \neq \ell'$

$= \frac{2}{2\ell+1} \quad \ell = \ell'$

Since boundary value pro sol^n unique, can use any method to get it.

~~E.g. potential due~~