Physics 829: Problem Set 5

Due Wednesday, May 13, 2009 at 11:59 P. M.

Each problem is worth 10 pts, unless otherwise stated.

1. As discussed in class, the operator describing the vector potential $\mathbf{A}(\mathbf{x},t)$ for an electromagnetic field is

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^2}{\omega_k}}\left[\mathbf{a}^\dagger_{\mathbf{k},\lambda}\exp(-i\mathbf{k}\cdot\mathbf{x})\hat{\epsilon}_{\mathbf{k},\lambda} + a_{\mathbf{k},\lambda}\exp(i\mathbf{k}\cdot\mathbf{x})\hat{\epsilon}_{\mathbf{k},\lambda}\right] \quad (1)$$

where $a_{\mathbf{k},\lambda}$ and $a^\dagger_{\mathbf{k},\lambda}$ are the lowering and raising operators for photons of wave vector $\mathbf{k}$ and polarization vector $\hat{\epsilon}_{\mathbf{k},\lambda}$. They satisfy the usual commutation relations for vibrational modes, i. e.,

$$\left[a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}\right] = \delta_{\mathbf{k},\mathbf{k}'}\hat{\epsilon}_{\mathbf{k},\lambda}\cdot\hat{\epsilon}_{\mathbf{k},\lambda}, \quad (2)$$

with all other commutators vanishing.

(a). Write down the electric field operator $\mathbf{E}(\mathbf{x},t) = -(1/c)\partial\mathbf{A}/\partial t$ and the magnetic induction operator $\mathbf{B}(\mathbf{x},t() = \nabla\times\mathbf{A}$, ;in terms of the various raising and lowering operators.

(b). Calculate nine commutators $[B_i(\mathbf{x}',t), E_j(\mathbf{x},t)]$, where i and j represent Cartesian components. You do not have to evaluate the sums over $\mathbf{k}$ which turn up in these commutators. What does this result say about the simultaneous measurability of $\mathbf{B}$ and $\mathbf{E}$? Note: the time dependence of these fields resides in the various $a^\dagger$'s and a's. What is the commutator $\sum_{i=1}^3[B_i(\mathbf{x}',t), E_i(\mathbf{x},t)]$?

2. Calculate $\langle\mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x}',t)\rangle$, i. e., the expectation value of the dot produce of the electric field at two points in the vacuum (no photons), at the same time. Note: in this case, the sum over $\mathbf{k}$ can be evaluated and you should evaluate it.

3. (20 pts.) Complete the calculation sketched out in class - that is, calculate the rate of spontaneous emission (in $\sec^{-1}$) from a hydrogen atom, initially in the state $|210\rangle$, to a final state $|100\rangle$. Here the numbers denote the radial quantum number n, the orbital quantum numbe ℓ, and

the azimuthal quantum number m. Neglect spin-orbit coupling and the spin degrees of freedom. Assume that the mass of the hydrogen atom is infinite, so that there is no recoil momentum when the photon is emitted. Make the electric dipole approximation, as discussed in class.

4. Consider an electron described by a Hamiltonian $p^2/(2m) + V(\mathbf{x})$. Let the eigenstates and eigenvalues of the Schrödinger equation be given by $|i\rangle$ and E_i respectively. Show that

$$\frac{i\hbar}{m}\langle I|\mathbf{p}|F\rangle = (E_F - E_I)\langle I|\mathbf{x}|F\rangle. \tag{3}$$

This result connects the momentum matrix element to the matrix element of the dipole operator $\mathbf{P} = -e\mathbf{x}$.

Note that $\mathbf{P}$ (the dipole operator) is not the same as the momentum operator $\mathbf{p}$.