Clarification of Derivation of Born Approximation from Fermi Golden Rule

Although the derivation of the Born approximation from the Fermi Golden Rule is basically correct, there are a couple of points which could use clarification. The basic Golden Rule formula is

$$w_{if} = \frac{2\pi}{\hbar}|\langle f|V|i\rangle|^2\rho_f. \tag{1}$$

Here w_{if} is the transition rate from an initial state $|i\rangle$ to a group of final states of the same energy, i. e., $E_f = E_i$, and we are considering only elastic scattering.

Now the matrix element will, in general, depend not only on the final energy but also on the angle between initial and final wave vectors. Therefore, we will define $w_{if}d\Omega/(4\pi)$ to be the transition rate from $|i\rangle$ to a group of final states of the same energy and with wave vector $\mathbf{k}_f$ within a particular solid angle element $d\Omega$. We also define $\rho_f d\Omega$ to be the number of states per unit energy such that the final wave vector lies within solid angle element $d\Omega$. This means that ρ_f is the density of states per unit energy per unit solid angle, and it equals the one calculated in class, multiplied by $d\Omega/(4\pi)$, or

$$\rho_f d\Omega = \frac{\bar{V}}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_f^{1/2}\frac{d\Omega}{4\pi}. \tag{2}$$

Thus, the rate for transitions into solid angle $d\Omega$ is

$$w_{if}d\Omega/(4\pi) = \frac{2\pi}{\hbar}|\langle \mathbf{p}_f|V|\mathbf{p}_i\rangle|^2\frac{\bar{V}}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_f^{1/2}\frac{d\Omega}{4\pi} \tag{3}$$

where $w_{if}d\Omega/(4\pi)$ represents the transition rate into solid angle element $d\Omega$.

To turn this into a differential cross section, we have to divide by the magnitude of the incident probability current density, which is $\hbar|\mathbf{k}_i|/[m\bar{V}]$, and by $d\Omega$. Thus, defining $|\mathbf{k}_i| = |\mathbf{k}_f| = k$, we have

$$\frac{d\sigma}{d\Omega} = \frac{w_{if}}{\hbar k/[m\bar{V}]} \tag{4}$$

or (after some algebraic simplification as discussed in the notes)

$$\frac{d\sigma}{d\Omega} = \left|\frac{m}{2\pi\hbar^2}\int e^{-i\mathbf{q}\cdot\mathbf{r}}V(\mathbf{r})d^3r\right|^2. \tag{5}$$