

Physics 829: Problem Set 6

Due Wednesday, May 15 at 11:59 P.M.

1. (25 pts.) **Linear and angular momentum of a distribution of electromagnetic fields in vacuum.**

(a). In classical electrodynamics, the momentum of a distribution of electromagnetic fields in vacuum is given by

$$\mathbf{P} = \frac{1}{4\pi c} \int d^3x \mathbf{E} \times \mathbf{B} \quad (1)$$

where the integration is over all space. Starting from the relation obtained in class,

$$\mathbf{A}(\mathbf{x}, t) = \sum_{\mathbf{k}, \lambda} \sqrt{\frac{2\pi\hbar c^2}{\omega_{\mathbf{k}}}} \left[a_{\mathbf{k}, \lambda} \mathbf{u}_{\mathbf{k}, \lambda}(\mathbf{x}) + a_{\mathbf{k}, \lambda}^\dagger \mathbf{u}_{\mathbf{k}, \lambda}^*(\mathbf{x}) \right], \quad (2)$$

where $\mathbf{u}_{\mathbf{k}, \lambda} = \hat{\mathbf{e}}_{\mathbf{k}, \lambda} e^{i\mathbf{k} \cdot \mathbf{x}} / \sqrt{V}$ and V is the volume of the system, obtain an expression for \mathbf{P} and show that

$$\mathbf{P} = \sum_{\mathbf{k}, \lambda} \hbar \mathbf{k} \left(n_{\mathbf{k}, \lambda} + \frac{1}{2} \right), \quad (3)$$

where $n_{\mathbf{k}, \lambda} = a_{\mathbf{k}, \lambda}^\dagger a_{\mathbf{k}, \lambda}$ is an operator describing the number of photons in state \mathbf{k} with polarization λ . Hence, the momentum of one such photon is $\hbar \mathbf{k}$.

(b). Classically, the angular momentum \mathcal{L} of a distribution of electromagnetic fields in vacuum is given by

$$\mathcal{L} = \frac{1}{4\pi c} \int d^3x \mathbf{x} \times (\mathbf{E} \times \mathbf{B}), \quad (4)$$

where the integration is over all space. Show that for fields localized to a finite region of space, provided that the magnetic field is eliminated in

favor of the vector potential \mathbf{A} , the angular momentum can be written in the form

$$\mathcal{L} = \frac{1}{4\pi c} \int d^3x \left[\mathbf{E} \times \mathbf{A} + \sum_{i=1}^3 E_i (\mathbf{x} \times \nabla) A_i \right]. \quad (5)$$

The first term is sometimes identified with the “spin” of the photon and the second with its “orbital” angular momentum (because of the present of the angular momentum operator $\mathbf{L}_{op} = -i\mathbf{x} \times \nabla$). (Note: this part is purely classical.)

(c). Show that \mathcal{L}_{spin} , the spin part of this angular momentum, is given by

$$\mathcal{L}_{spin} = \hbar \sum_{\mathbf{k}, \lambda} (n_{\mathbf{k},+} - n_{\mathbf{k},-}), \quad (6)$$

where $n_{\mathbf{k},\pm}$ is the number of photons of wave vector \mathbf{k} and polarization vector $\hat{\epsilon}_{\pm} = (1/\sqrt{2})(\hat{\epsilon}_1 \pm \hat{\epsilon}_2)$, and $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ are unit vectors describing the two possible linear polarizations of a photon of wave vector \mathbf{k} . Note: for this problem, you should use the expansion given above for $\mathbf{A}(\mathbf{x}, t)$ but, in summing over polarizations, use the two circular polarizations given above rather than the linear polarization vectors. Also, note that $\hat{\epsilon}_+^* = \hat{\epsilon}_-$.

Thus, the “spin” angular momentum of a photon is $\pm\hbar$. Hence, the photon is sometimes described as a spin-1 particle (except that the $S_z = 0$ part of the triplet is missing).

2. **Dipole-dipole term in hyperfine interaction** One term in the hyperfine interaction hamiltonian, as discussed in class, is

$$H_{dip-dip} = \frac{1}{r^3} \left(\mathbf{M}_S \cdot \mathbf{M}_I - \frac{3(\mathbf{r} \cdot \mathbf{M}_S)(\mathbf{r} \cdot \mathbf{M}_I)}{r^2} \right), \quad (7)$$

where $\mathbf{M}_S = g_e \mu_e \mathbf{S}$ and $\mathbf{M}_I = g_n \mu_n \mathbf{I}$ are the magnetic moment operators of the electronic and nuclear spins, g_e and g_n are the electronic and nuclear spin g-factors, μ_e and μ_n are the Bohr magnetons corresponding to the electronic and nuclear spins, and \mathbf{S} and \mathbf{I} are the electronic and nuclear spin angular momentum. Assume that the nucleus has spin 1/2

Show that $\langle \psi | H_{dip-dip} | \psi \rangle = 0$ if the electronic wave function $\langle \mathbf{r} | \psi \rangle$ is spherically symmetric.

3. **Zeeman effect in the 1s ground state.** The Hamiltonian for a hydrogen 1s electron, including the hyperfine interaction with the proton and also an applied magnetic field, may be written

$$H = \mathcal{A} \mathbf{S}_1 \cdot \mathbf{S}_2 + \omega_1 S_{1z} + \omega_2 S_{2z}. \quad (8)$$

Here \mathbf{S}_1 and \mathbf{S}_2 represent the spin angular momenta of the electron and the proton (each of which is spin 1/2), \mathcal{A} is the proton-electron hyperfine interaction, $\omega_1 = -\gamma_1 B_0$, $\omega_2 = -\gamma_2 B_0$, B_0 is the applied magnetic field (assumed to be in the z direction) and γ_1 and γ_2 are the gyromagnetic ratios of the electron and proton.

Calculate the four energy eigenvalues, and corresponding eigenstates, of this hamiltonian. What are the limiting forms of the eigenstates in (a) zero field, and (b) very strong field?