Physics 829: Problem Set 5

Due Wednesday, May 7 at 11:59 PM

Each problem is worth 10 pts. unless otherwise stated.

1. As discussed in class, the operator describing the vector potential $\mathbf{A}(\mathbf{x}, t)$ for an electromagnetic field is

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},\lambda} \sqrt{\frac{2\pi\hbar c^2}{\omega_k}} \left[a^\dagger_{\mathbf{k}\lambda} \exp(-i\mathbf{k}\cdot\mathbf{x})\hat{\epsilon}_\lambda + a_{\mathbf{k}\lambda} \exp(i\mathbf{k}\cdot\mathbf{x})\hat{\epsilon}_\lambda\right], \quad (1)$$

where $a_{\mathbf{k}\lambda}$ and $a^\dagger_{\mathbf{k}\lambda}$ are the lowering and raising operators for photons of wave vector $\mathbf{k}$ and polarization vector $\hat{\epsilon}_\lambda$, and satisfy the usual commutation relations for vibrational modes, i. e.,

$$\left[a_{\mathbf{k}\lambda}, a^\dagger_{\mathbf{k}'\lambda'}\right] = \delta_{\mathbf{k},\mathbf{k}'}\hat{\epsilon}_\lambda \cdot \hat{\epsilon}_{\lambda'}, (2)$$

with all other commutators vanishing.

(a) Write down the electric field operator $\mathbf{E}(\mathbf{x}, t) = -(1/c)\partial\mathbf{A}/\partial t$ and magnetic induction operator $\mathbf{B}(\mathbf{x}, t) = \nabla \times \mathbf{A}$, in terms of the various raising and lowering operators.

(b). Calculate the commutator $[\mathbf{B}(\mathbf{x}', t'), \mathbf{E}(\mathbf{x}, t)]$. What does this say about the simultaneous measurability of $\mathbf{B}$ and $\mathbf{E}$? Note: the time-dependence of these fields resides in the various a's and $a^\dagger$'s.

2. Calculate $\langle \mathbf{E}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}'t)\rangle$, i. e., the expectation value of the dot product of the electric field at two points in the vacuum (no photons), at the same time.

3. (20) Complete the calculation sketched out in class - that is, calculate the rate of spontaneous emission (in $\sec^{-1}$) from a hydrogen atom, initially in the state $|210\rangle$, to a final state $|100\rangle$. Here, the numbers denote the radial quantum number n, the orbital quantum number

ℓ, and the azimuthal quantum number m. Neglect spin-orbit coupling and the spin degrees of freedom. Assume that the mass of the hydrogen atom is infinite, so that there is no recoil momentum when the photon is emitted. Make the electric dipole approximation, as discussed in class.

4. Consider an electron described by a Hamiltonian $p^2/(2m) + V(\mathbf{x})$. Let the eigenstates and eigenvalues of the Schrödinger equation be given by $|i\rangle$ and E_i, respectively. Show that

$$\frac{i\hbar}{m}\langle i|\mathbf{p}|f\rangle = (E_f - E_i)\langle i|\mathbf{x}|f\rangle. \tag{3}$$

This result connects the momentum matrix element to the matrix element of the electric dipole operator.