Physics 828: Problem Set VIII

Dr. Stroud

Due Wednesday, March 11, 2009 by 11:59 P. M.

Each problem is worth 10 pts. unless otherwise specified.

1. Shankar, exercise 18.4.2.
2. Shankar, exercise 18.4.4.
3. (20 pts.) A hydrogen atom is initially in its 1s state. A uniform electric field $\mathbf{E}\cos(\omega t)$ is applied to the atom. Calculate the ionization rate as a function of ω. You may assume that the final states are free-particle states, and also that the proton is infinitely massive.

   In this problem, the calculation is difficult if you do not assume that ka is small compared to unity (here k is the wave vector of the final, free-electron state and a is the linear dimension of the atomic 1s state). Therefore, you should make this assumption in your calculation, and just calculate the ionization rate to lowest non-vanishing power in ka.
4. (20pts.) **Time-Dependent Schrödinger equation, two-level system, resonant perturbation.** In this problem, you will obtain a (nearly) exact solution to the time-dependent Schrödinger equation for a certain special case.

   As was discussed in class, the solution of the time-dependent Schrödinger equation can be written in the form

$$|\psi(t)\rangle = \sum_n c_n(t)\exp(-iE_nt/\hbar)|\phi_n\rangle, \tag{1}$$

where the $|\phi_n\rangle$ are a complete orthonormal basis and the $c_n(t)$'s are a set of time-dependent coefficients. The Hamiltonian of the system is assumed to be $H = H_0 + W(t)$, where H_0 is a time-independent operator and $W(t)$ is a time-dependent perturbation. The energies E_n are the eigenvalues of the unperturbed Schrödinger equation, i. e., $H_0|\phi_n\rangle = E_n|\phi_n\rangle$.

We now assume that there are only two states in the basis, which we denote $|1\rangle$ and $|2\rangle$. We also assume that $H_0|1\rangle = E_1|1\rangle$ and $H_0|2\rangle = E_2|2\rangle$. Finally, we assume that $W(t) = W\sin(\omega t)$, where W is time-independent. Make the assumption that $\langle 1|W|1\rangle = \langle 2|W|2\rangle = 0$.

(a). Explain why $W_{21} = W_{12}^*$.

(b). Show that $c_1(t)$ and $c_2(t)$ satisfy the set of equations

$$\begin{aligned} i\hbar\dot{c}_1(t) &= \frac{1}{2i}\left[e^{i(\omega-\omega_{21})t} - e^{-i(\omega+\omega_{21})t}\right] W_{12}c_2(t) \\ i\hbar\dot{c}_2(t) &= \frac{1}{2i}\left[e^{i(\omega+\omega_{21})t} - e^{-i(\omega-\omega_{21})t}\right] W_{21}c_1(t). \end{aligned} \tag{2}$$

Here the dot denotes a time derivative and $\omega_{21} = (E_2 - E_1)/\hbar$.

(c). Now, assume that $\omega = \omega_{21}$ ("resonant perturbation"). In that case, the coefficients of $c_1(t)$ and $c_2(t)$ are each the sum of two terms: a time-independent term, and a rapidly varying time-dependent term. Show that if you neglect the time-dependent terms, you can obtain the following equation for $c_1(t)$:

$$\ddot{c}_1(t) = -\frac{1}{4\hbar^2}|W_{12}|^2 c_1(t). \tag{3}$$

(d). Solve this equation under the assumption that $c_1(0) = 1$, and show that the probability of finding the system in state 2 is

$$P_2(t) = \sin^2\left[|W_{21}|t/(2\hbar)\right]. \tag{4}$$

Thus, $P_2(t)$ oscillates in time with an angular frequency $W_{21}/(2\hbar)$. These oscillations are called *Rabi oscillations.*