Physics 828: Problem Set 5

Due Monday, February 16, 2009 at 11:59 PM

1. Consider the one-dimensional harmonic oscillator with a quartic perturbation. The Hamiltonian is then given by

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 + \lambda x^4. \quad (1)$$

where k is the spring constant and λ is a parameter governing the size of the perturbing potential.

(a) Calculate the perturbed energy levels to first order in λ. Hint: use the raising and lowering operators for the harmonic oscillator.

(b). For a given λ, estimate the quantum number n where the first-order perturbation becomes inaccurate.

(c). Write down *formally* the second order term to the energy correction. Explain which matrix elements in the sum will be non-zero, and why.

(d). What is the sign of the second-order correction for the ground state (n = 0)?

2. Consider a spin-1/2 particle in a time-independent external magnetic field $B_x\hat{x} + B_z\hat{z}$. Assume that the Hamiltonian is $H = -\gamma\mathbf{S}\cdot\mathbf{B}$, where $\gamma > 0$ and $B_x > 0$.

(a). Find the exact energy eigenvalues and corresponding eigenstates of the Hamiltonian. Expressthe eigenstates in the basis of eigenstates of S_z, denoted $|\uparrow\rangle$ and $|\downarrow\rangle$.

(b). Now assume that $|B_x| \ll |B_z|$. The Hamiltonian can now be expressed as the sum of an unperturbed part $H_0 = -\gamma S_z B_z$ and a perturbation $H_1 = -\gamma S_x B_x$. Calculate the energy eigenvalues through second order in H_1, and the correpsonding eigenstates through first order in H_1, and compare these to the exact results for these quantities.

3. (The following is a one-dimensional problem.) Suppose an electron of energy E experiences the following potential in an electric field $\mathcal{E}$:

$$\begin{aligned} V(z) &= 0 \qquad\qquad (z < 0) \\ V(z) &= \Phi - e\mathcal{E}z \qquad\qquad (z > 0). \end{aligned} \tag{2}$$

Here $\Phi > 0$ is called the work function, and we use the convention that $e > 0$.

Use the WKB approximation to calculate the transmission coefficient, as a function of the electric field.

4. Prove the Thomas-Reiche-Kuhn sum rule,

$$\sum_{n' \neq n} (E_n - E_{n'}) |\langle n'|x|n\rangle|^2 = -\frac{\hbar^2}{2m}. \tag{3}$$

Here $|n\rangle$ and E_n are eigenstates and energy eigenvalues of a (one-dimensional) Hamiltonian $H = p^2/(2m) + V(x)$, i. e. $H|n\rangle = E_n|n\rangle$.