Conservation of Energy for a System of N Particles

Here is a much simpler way to derive conservation of energy than the one I tried to use in class.

Let us assume that we have a collection of N particles, located at $\mathbf{r}_1$, $\mathbf{r}_2$,... $\mathbf{r}_N$, and having masses m_1, m_2,..., m_N. We write the α^{th} Cartesian component of the position vector of the ith particle as $x_{i\alpha}$ ($\alpha = 1,\ 2,\ 3$). Thus, the ith position vector is $\mathbf{r}_i = (x_{i1}, x_{i2}, x_{i3})$

We also assume that the forces acting on the particles can be derived from a potential function. Specifically, we assume that the force on the ith particle can be written

$$\mathbf{F}_i = -\nabla_i V(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N). \tag{1}$$

Here V is a function of the 3N variables $x_{i\alpha}$ (that is, three Cartesian coordinates for each of the N particles). The operator ∇_i is a three-component vector and can be written

$$\nabla_i = \left(\frac{\partial}{\partial x_{i1}}, \frac{\partial}{\partial x_{i2}}, \frac{\partial}{\partial x_{i3}} \right). \tag{2}$$

Newton's second law then takes the form

$$\frac{d}{dt}(m_i \mathbf{v}_i) = -\nabla_i V(\mathbf{r}_1, ...\mathbf{r}_N). \tag{3}$$

Now multiply both sides of this equation by $\mathbf{v}_i$ and sum over i to get

$$\sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \frac{d}{dt}\mathbf{v}_i = -\sum_{i=1}^{N} \mathbf{v}_i \cdot \nabla_i V(\mathbf{r}_1, ...\mathbf{r}_N), \tag{4}$$

where we assume that the m$_i$'s are independent of time.

Consider first the left-hand side of eq. (4). We can use the relation

$$\mathbf{v}_i \cdot \frac{d}{dt}\mathbf{v}_i = \frac{d}{dt}\frac{1}{2}\mathbf{v} \cdot \mathbf{v} \tag{5}$$

to write the left-hand side as

$$\frac{d}{dt}\sum_{i=1}^{N}\frac{1}{2}m_i v_i^2, \tag{6}$$

where $v_i^2 = \mathbf{v}_i \cdot \mathbf{v}_i$.

Now consider the right-hand side of eq. (4). Writing $\mathbf{v}_i = d\mathbf{r}_i/dt$, we can write the right-hand side as

$$-\sum_{i=1}^{N}\nabla_i V(\mathbf{r}_1, ..., \mathbf{r}_N)\cdot\frac{d\mathbf{r}_i}{dt} = -\sum_{i=1}^{N}\sum_{\alpha=1}^{3}\frac{\partial V}{\partial x_{i\alpha}}\frac{dx_{i\alpha}}{dt} = -\frac{dV}{dt}, \tag{7}$$

where the last equality follows from the chain rule in calculus of several variables. That is, the potential energy V depends on time only implicitly, through the variables $x_{i\alpha}$, i. e., the components of the position vectors.

Setting expressions (6) and (7) equal, we obtain

$$\frac{dT}{dt} = -\frac{dV}{dt}, \tag{8}$$

where

$$T = \sum_{i=1}^{N}\frac{1}{2}m_i v_i^2. \tag{9}$$

Or, simplifying, we get

$$\frac{dE}{dt} = 0 \tag{10}$$

where

$$E = T + V \tag{11}$$

is the *total energy.* Thus, if the force acting on the ith particle can be obtained as the gradient of a potential, as in eq. (1), the total energy, as defined in eq. (11), is a constant of the motion.