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Theory of fluctuations in a network of parallel superconducting wires

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ABSTRACT

We show how the partition function of a network of parallel superconducting wires weakly coupled together by the proximity effect, subjected to a vector potential along the wires, can be mapped onto N-distinguishable two dimensional quantum-mechanics problem with a perpendicular imaginary magnetic field. Then, we show, using a mean field approximation, that, for a given coupling, there is a critical temperature for onset of inter-wire phase coherence. The transition temperature T_c is plotted on both cases for non-magnetic and a magnetic field perpendicular to the wires.

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1. Introduction

There has been considerable recent interest in thin wires that undergo transitions into an ordered state, such as superconducting or ferromagnetic. For example, experiments have suggested that single-walled carbon nanotubes (which have diameters of only about 4 Å) are superconducting with a transition temperature of 15 K [1,2]. Because these tubes are so thin, they behave very much like one-dimensional superconductors. It was therefore proposed [1] that they could be described by a complex order parameter $\psi(z)$ which varies only in one dimension, say the z-direction, i.e. along the tube. $\psi(z)$ might represent the complex energy gap, or, in a different normalization, it could represent the condensate wave function in a BCS superconductor.

Moreover, there have been many experiments for investigating superconductivity on nanowires. Ropes of carbon nanotubes between superconducting electrodes can show superconductivity due to the proximity effect of the electrodes [3–5]. Furthermore, superconductivity on carbon nanowires connected to normal contacts, has been observed [6,7]. On the other hand, superconductivity of nanowires of Zn or Sn has been investigated [8–10].

Fluctuations are, of course, especially important in one-dimensional systems. It was shown many years ago by Scalapino et al. [11] that *classical* fluctuations in one dimension could be treated *exactly*, within the context of a Ginzburg–Landau (GL) free energy functional. Their treatment involved mapping the GL functional onto a single-particle quantum mechanics problem, using an exact connection between the classical partition function and a path integral treatment of the quantum mechanics problem. These authors showed that classical fluctuations could give rise to a non-zero order parameter even above the GL transition temperature. This mapping was extended to treat Josephson-coupled thin wires [12,13].

However, in the mapping, the effect of a magnetic field was not included. In the present paper, we consider the case of a non-zero magnetic field perpendicular to the wires. We show that the partition function for this system maps onto a certain zero-temperature quantum mechanics problem in two dimensions with an effective imaginary perpendicular magnetic field. This reduces the solution of the partition function to solving a certain problem in non-Hermitian quantum mechanics.

The non-Hermitian problem in physics is not new. Nonequilibrium processes can be described by non-Hermitian Liouville operators [14–16]. The non-Hermitian quantum mechanics has also been used to study the pinning of magnetic flux lines in high temperature superconductors [17–20].

The remainder of this paper is organized as follows. In Section 2, we describe our formalism and mapping. In Section 3, we give our numerical results, including approximate phase diagrams. This is followed by a concluding discussion and an outline of possible future research.

2. Formalism

2.1. Mapping to a quantum mechanics problem for interacting superconducting wires when $\vec{B}$ is perpendicular to the wires

Let us consider a network of N parallel superconducting wires in a non-zero vector potential. We assume, for convenience, that

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these wires all have the same GL parameters, though the formalism can easily be generalized to the case when the parameters are different. Then the partition function can be written as a functional integral over the N complex order parameters $\psi_1(z_1), \ldots, \psi_N(z_N)$:

$$Z = \int \mathscr{D}\psi_1(z_1)\ldots\mathscr{D}\psi_N(z_N)\exp\{-\beta F[\psi_1(z_1),\ldots\psi_N(z_N)]\}. \tag{1}$$

We assume that the free energy functional is the sum of two parts: a single-wire term F_s and a term describing inter-wire interactions, which we denote F_{int}. The single-wire term will just be the sum of GL free energies for each wire:

$$F_s = \sum_{i=1}^{N} F_{GL}[\psi_i(z_i)]. \tag{2}$$

Here,

$$F_{GL}[\psi_i(z_i)] = \int_0^{z_{max}} \left[\frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla - \frac{e^*\vec{A}}{c}\right)\psi(z)\right|^2 + \alpha|\psi(z)|^2 + \gamma|\psi(z)|^4 + \frac{HB}{8\pi}\Sigma\right]dz, \tag{3}$$

where α, γ, and m^* are material-dependent (and possibly temperature-dependent) coefficients. Commonly, it is assumed that γ is positive and that $\alpha = \alpha'(T - T_c)$, where T is the temperature, T_c is the critical temperature, and α' is greater than zero. In the last term, Σ is the cross-sectional area of the sample; for a sufficiently thin (effectively one-dimensional) wire, we may ignore this term. For the interaction term, we assume a form similar to that used by Lawrence and Doniach for interacting superconducting layers [21], namely

$$F_{int} = \sum_{\langle ij\rangle}\int_0^{z_{max}} K_{ij}|\psi_i(z) - \psi_j(z)|^2. \tag{4}$$

where z_{max} is the length of the wires. Basically, we are assuming that there is a Josephson coupling of strength K_{ij} between different wires, but at the same point along the length, z.

We will be interested in treating a magnetic field perpendicular to the wires. Thus, we choose a gauge such that the vector potential is parallel to the superconducting wires, has only z component and independent of z for simplicity. When a wire is a loop, a vector potential is related to the total flux Φ through the loop, $A_z = \Phi/z_{max}$.

In this case, using $\psi_i(z) = \psi_{iR}(z) + i\psi_{iI}(z)$, F_s and F_{int} take the forms

$$F_s = \sum_i \int_0^{z_{max}} \left[\frac{\hbar^2}{2m^*}|\psi_i'|^2 - \frac{e^*\hbar A_z}{m^* c}(\psi_{iR}\psi_{iI}' - \psi_{iR}'\psi_{iI}) + \left\{\alpha + \frac{1}{2m^*}\left(\frac{e^*}{c}\right)^2 A_z^2\right\}|\psi_i|^2 + \gamma|\psi_i|^4\right]dz, \tag{5}$$

and

$$F_{int} = \sum_{\langle ij\rangle}\int_0^{z_{max}} K_{ij}(|\psi_i(z)|^2 + |\psi_j(z)|^2 - 2(\psi_{iR}\psi_{jR} + \psi_{iI}\psi_{jI}))dz, \tag{6}$$

where $\psi'(z) = d\psi(z)/dz$. Finally, the partition function takes the form

$$Z = \int \phi\mathscr{D}\psi_{iR}\mathscr{D}\psi_{iI}\exp(-\beta F[\psi_{iR},\psi_{iI}]). \tag{7}$$

We now show that Eqs. (5)–(7) for Z are actually equivalent to a *quantum mechanical* problem of a N distinguishable particles in N distinct quantum wells in two dimensions in the presence of an effective magnetic field $\mathbf{B}_{eff}$ which is perpendicular to the plane of this effective two-dimensional problem. In order to simplify our argument, we consider the case of single particle with mass m and a charge e^* subjected to a 2D potential $V(x,y)$. The density matrix of a two-dimensional system, using ψ^I and ψ^F as the values of the wave functions at initial and final time, can be written as a path integral of the form[22,23]

$$\langle\psi^F|e^{-S/\hbar}|\psi^I\rangle = \int_{\psi^I}^{\psi^F}\mathscr{D}x(\tau)\mathscr{D}y(\tau)\exp\left\{-\frac{1}{\hbar}S[x(\tau),y(\tau)]\right\}, \tag{8}$$

where

$$S = \int_0^{\beta_{eff}\hbar}\left[\frac{m}{2}(x'^2 + y'^2) + V(x,y) - i\frac{e^*}{c}\vec{A}_{eff}\cdot\vec{v}\right]d\tau, \tag{9}$$

where $\beta_{eff} = 1/k_B T_{eff}$, $x' = dx/d\tau$, $y' = dy/d\tau$, and $\vec{v}$ is a two-component vector with components $(dx/d\tau,\ dy/d\tau)$. For the given $\vec{B}_{eff} = B_{eff}\hat{z}$ with the gauge

$$\vec{A}_{eff} = \frac{B_{eff}}{2}(x\hat{y} - y\hat{x}), \tag{10}$$

this S becomes

$$S = \int_0^{\beta_{eff}\hbar} d\tau\left[\frac{m}{2}(x'^2 + y'^2) + V(x,y) - i\frac{e^*B_{eff}}{2c}(xy' - yx')\right]. \tag{11}$$

This is similar equation to the expression (7) for the partition function of a single superconducting wire, provided we identify the proper correspondences between terms in the free energy of the superconducting wire and the equivalent quantum-mechanical problem.

The interaction term between the wires can also be translated to an equivalent quantum-mechanical problem, and we can obtain a complete correspondence between the 1D classical problem and the 2D quantum-mechanical one. In order to simplify this mapping, we use the suitable variables: $\tilde{\psi}_{ix} = \xi_0^{3/2}\psi_{iR}$, and $\tilde{\psi}_{iy} = \xi_0^{3/2}\psi_{iI}$, where ξ_0 is related to the superconducting coherence length. With these definitions, we can make the identifications shown in Table 1.

We find that the magnetic field has two effects: (i) it determines an effective perpendicular magnetic field in which the equivalent quantum-mechanical particle moves and (ii) it changes the quadratic part of the effective potential. The Hamiltonian for the analogous quantum problem for many interacting wires is

$$H = \sum_{i=1}^{N}\left[\frac{1}{2m}\left(p_{ix} + \frac{e^*B_{eff}}{2c}y\right)^2 + \frac{1}{2m}\left(p_{iy} - \frac{e^*B_{eff}}{2c}x\right)^2 + V_i(\vec{\rho}_i)\right] + \sum_{<ij>}J_{ij}|\vec{\rho}_i - \vec{\rho}_j|^2, \tag{12}$$

where p_{ix} and p_{iy} are the momentum operators of the x and y components of the ith particle, respectively.

Table 1
Correspondences in the mapping between quantum mechanical (Q.M.) and superconducting (S.C.) problems described in the text. In each case, the left-hand variable corresponds to the quantum mechanical problem and the right-hand variable corresponds to the problem of parallel superconducting wires. The various symbols are defined in the text.

Q.M.	S.C.
τ	$\frac{\beta\hbar}{\xi_0}z$
$\vec{\rho}_i = \{x_i(u), y_i(u)\}$	$\vec{\psi}_i = \{\tilde{\psi}_{ix}(z), \tilde{\psi}_{iy}(z)\}$
E	$F\frac{\xi_0}{z_{max}}$
β_{eff}	$\beta z_{max}/\xi_0$
$V_i(x_i,y_i)$	$\left(\frac{\alpha}{\xi_0^2} + \frac{1}{2m^*\xi_0^2}\left(\frac{e^*A_z}{c}\right)^2\right)\lvert\tilde{\psi}_i\rvert^2 + \frac{\gamma}{\xi_0^5}\lvert\tilde{\psi}_i\rvert^4$
m	$\frac{\hbar^4\beta^2}{m^*\xi_0^4}$
B_{eff}	$-i\frac{2\hbar^2 A_z\beta}{m^*\xi_0^3}$
J_{ij}	$\frac{K_{ij}}{\xi_0^2}$

2.2. Probability distribution of the order parameter

As a first application, we consider the probability distribution of the superconducting order parameter, which corresponds to the probability distribution of the particles in the quantum-mechanical problem. In order to simplify our discussion, we first consider single wire case. The probability distribution function of the order parameter can be defined as

$$P(\vec{\rho}(\tau)) = \frac{1}{Z}\langle\psi^F|e^{-\frac{H}{\hbar}(L_\tau-\tau)}|\vec{\rho}(\tau)\rangle\langle\vec{\rho}(\tau)|e^{-\frac{H}{\hbar}\tau}|\psi^I\rangle, \quad (13)$$

where $Z = \langle\psi^F\left|e^{-\frac{H}{\hbar}L_\tau}\right|\psi^I\rangle$ and $|\psi^I\rangle$ represents the boundary condition at $\tau = 0$ and $\langle\psi^F|$ represents the boundary condition at $\tau = L_\tau$. Using the eigenstates of the Hamiltonian, $H|n\rangle = E_n|n\rangle$, the probability can be written as

$$P(\vec{\rho}(\tau)) = \frac{1}{Z}\sum_{m,n}\langle\psi^F|m\rangle\langle m|\vec{\rho}(\tau)\rangle\langle\vec{\rho}(\tau)|n\rangle\langle n|\psi^I\rangle e^{-\frac{E_m}{\hbar}(L_\tau-\tau)}e^{-\frac{E_n}{\hbar}\tau} \quad (14)$$

with

$$Z = \sum_n\langle\psi^F|n\rangle\langle n|\psi^I\rangle e^{-\frac{E_n}{\hbar}\tau}. \quad (15)$$

Explicitly, the expectation value of operator, ρ at the distance τ from the bottom of the wires is given by

$$\langle\hat{\rho}\rangle_\tau = \frac{1}{Z}\langle\psi^F\left|e^{-\frac{H}{\hbar}(L_\tau-\tau)}\int d\vec{\rho}(\tau)\right|\vec{\rho}(\tau)\rangle\rho\langle\vec{\rho}(\tau)\left|e^{-\frac{H}{\hbar}\tau}\right|\psi^I\rangle, \quad (16)$$

where $\hat{\rho}|\vec{\rho}\rangle = \rho|\vec{\rho}\rangle$.

In the case of periodic boundary conditions, our problem can be simplified. If $\psi^F = \psi^I$, then the probability distribution for the order parameter corresponds to summing over all possible configurations consistent with this condition and can be written

$$\begin{aligned}P(\vec{\rho}(\tau)) &= \frac{1}{Z}\sum_{m,n}\sum_I\langle\psi^I|m\rangle\langle m|\vec{\rho}(\tau)\rangle\langle\vec{\rho}(\tau)|n\rangle\langle n|\psi^I\rangle e^{-\frac{E_m}{\hbar}(L_\tau-\tau)}e^{-\frac{E_n}{\hbar}\tau} \\ &= \frac{1}{Z}\sum_n\langle n|\vec{\rho}(\tau)\rangle\langle\vec{\rho}(\tau)|n\rangle e^{-\frac{E_n}{\hbar}L_\tau},\end{aligned}$$

where $Z = \sum_n\sum_I\langle\psi^I|n\rangle\langle n|\psi^I\rangle e^{-\frac{E_n}{\hbar}L_\tau} = \sum_n e^{-\frac{E_n}{\hbar}L_\tau}$. So, if the wire is actually in the form of a loop, which means the boundary conditions $\psi(0) = \psi(z_{max})$, our problem corresponds to this statistical mechanics.

For a single wire, in the case of the periodic boundary conditions, we can understand the behavior of the order parameter qualitatively. The average gap in the GL problem (which we denote $\widetilde{\Delta}(t) = \langle\psi\left(t = \frac{T}{T_c^0}\right)\rangle$ corresponds to the mean distance $\langle\rho\rangle$ in the quantum-mechanical problem, i.e.

$$\langle\rho\rangle \to \widetilde{\Delta}(t), \quad (17)$$

where $\Delta(t) = \widetilde{\Delta}(t)/\xi_0^{3/2}$. For a temperature T much lower than the critical temperature T_c^0, the mean distance from the origin of the particle approaches the value predicted for the quantum problem in the limit of infinite mass, i.e. the value of ρ for which the quartic potential is a minimum. The function $\sqrt{1-t}$, is the classical solution, i.e., the solution in the case when thermal fluctuations in the GL case are negligible. These fluctuations do indeed become very small when $T \simeq 0$, because in this regime, the effective potential rises steeply above its minimum, and $\langle\rho\rangle$ becomes very close to the value that minimizes the GL free energy. When $\langle\rho\rangle$ has this value, the corresponding value for $\widetilde{\Delta}(t)$ is

$$\widetilde{\Delta}(t) = \sqrt{\tilde{\psi}_R{}^2 + \tilde{\psi}_I{}^2} = \sqrt{\frac{\alpha_0 T_c^0\xi_0^3}{2\gamma}}\sqrt{1-t} = \widetilde{\Delta}(0)g(t), \quad (18)$$

where $\widetilde{\Delta}(0)$ is the gap at $T = 0$. These considerations may suggest that we can approximate $g(t) = \widetilde{\Delta}(t)/\widetilde{\Delta}(0) = \sqrt{1-t}$.

2.3. Phase only model and mean-field approximation

The system of coupled parallel superconducting wires will undergo a phase transition into a phase-ordered state below a critical temperature T_c which is distinct from (and lower than) the single wire mean-field transition temperature T_c^0. To estimate the properties of this transition, we consider a simplified, "phase-only" version of the equivalent Schrödinger Eq. (12). We assume that the *magnitudes* ρ_i of the variables $\mathbf{x}_i$ are fixed at the values which minimize the single-wire GL free energy, i.e. $\rho_i \equiv \rho_0$, where ρ_0 is given by Eq. (18). All terms in the Hamiltonian involving $\partial/\partial\rho_i$ can be ignored in this phase-only model. The effective Hamiltonian (12) then becomes

$$\begin{aligned}H = &-\sum_i\frac{\hbar^2}{2m\rho_0^2}\frac{\partial^2}{\partial\phi_i^2} - \sum_i\frac{e^*B_{eff}}{2mc}\frac{\hbar}{i}\frac{\partial}{\partial\phi_i} \\ &+ 2\sum_{\langle ij\rangle}J_{ij}\rho_0^2(1-\cos(\phi_i-\phi_j)),\end{aligned} \quad (19)$$

where the double sum in the third term runs over distinct nearest neighbor pairs. When $B_{eff} = 0$, this is the well-known *quantum XY model*, which exhibits a quantum phase transition at a critical value of the ratio between the coefficients of the first and the third term.

Let us assume that $J_{ij} = J$ for nearest neighbor wires and $J_{ij} = 0$ otherwise. Then we can apply the mean field approximation to this Hamiltonian by replacing the third term according to the prescription

$$\cos(\phi_i-\phi_j) = 2\cos\phi_i\langle\cos\phi\rangle - \langle\cos\phi\rangle^2, \quad (20)$$

where we are supposing $\langle\sin\phi\rangle = 0$ because of the symmetry. Thus,

$$\begin{aligned}2\sum_{\langle ij\rangle}J\rho_0^2(1-\cos(\phi_i-\phi_j)) &= 2\sum_{\langle ij\rangle}J\rho_0^2\left(1-2\cos\phi_i\langle\cos\phi\rangle+\langle\cos\phi\rangle^2\right) \\ &= -4z_nJ\rho_0^2\langle\cos\phi\rangle\sum_i\cos\phi_i \\ &\quad + 2\sum_{\langle ij\rangle}J\rho_0^2(1+\langle\cos\phi\rangle^2) \\ &= -4z_nJ\rho_0^2\langle\cos\phi\rangle\sum_i\cos\phi_i \\ &\quad + 2z_nNJ\rho_0^2(1+\langle\cos\phi\rangle^2),\end{aligned}$$

where z_n is the number of nearest neighbors in the lattice (e. g. $z_0 = 4$ for a square lattice). Thus, the effective Hamiltonian corresponding to Eq. (19) becomes the following Schrödinger equation:

$$\begin{aligned}&\left\{\frac{-\hbar^2}{2m\rho_0^2}\frac{\partial^2}{\partial\phi_i^2} - \frac{e^*B_{eff}}{2mc}\frac{\hbar}{i}\frac{\partial}{\partial\phi_i} - 4z_n\rho_0^2J\langle\cos\phi\rangle\cos\phi_i + 2z_nJ\rho_0^2(1+\langle\cos\phi\rangle^2)\right\}\psi_n(\phi_i) \\ &= E_n\psi_n(\phi_i).\end{aligned} \quad (21)$$

In order to find the phase transition for given m, J, ρ_0, z_n, and B_{eff}, we solve this equation self-consistently for $\langle\cos\phi\rangle$, assuming a periodic boundary condition for $\psi_n(\phi)$. The mean field theory is defined by the self-consistency requirement on $\langle\cos\phi\rangle$:

$$\langle\cos\phi\rangle = \frac{\sum_n e^{-\beta_{eff}E_n}\langle\psi_n(\phi_i)|\cos\phi_i|\psi_n(\phi_i)\rangle}{\sum_n e^{-\beta_{eff}E_n}}. \quad (22)$$

For example, when the wires are sufficiently long that only the ground state contribution may be important, the self-consistency condition becomes

$$\langle\cos\phi\rangle = \langle\psi_0(\phi_i)|\cos\phi_i|\psi_0(\phi_i)\rangle. \quad (23)$$

These equations may be solved for $\langle\cos\phi\rangle$ and T_c, where the critical temperature can be determined by $\langle\cos\phi\rangle = 0$.

3. Results and Discussion

We have considered long-range phase coherence among wires in the bundle in order to see whether the phases on the wires are coherent and the bundle as a whole is superconducting or not. The self-consistent equation gives rise to a phase diagram exhibiting superconductivity, which can be defined as the greatest temperature and field such that $\langle\cos\phi\rangle$ takes on a non-zero value [24]. Here, we assume that the Josephson coupling is independent of a temperature. We consider the temperature dependence, $\sqrt{1-t}$ for ρ. In order to simplify our calculations, we consider only the case of periodic boundary conditions.

3.1. No magnetic field

We consider the following self-consistent equation, substituting $B_{eff} = 0$ for the differential Eq. (21),

$$\left(-\frac{\hbar^2}{2m\rho_0^2}\frac{\partial^2}{\partial\phi_i^2} - 4z_n\rho_0^2 J\langle\cos\phi\rangle\cos\phi_i + 2z_n J\rho_0^2(1+\langle\cos\phi\rangle^2)\right)\psi_n(\phi_i) = E_n\psi_n(\phi_i). \tag{24}$$

This equation can be reduced to the standard Mathieu equation [25], using $v = \phi/2$, $y(v) = \psi_n(\phi_i/2)$,

$$\frac{d^2y_n(v)}{dv^2} + (a_n - 2q\cos 2v)y_n(v) = 0, \tag{25}$$

where the characteristic value of the Mathieu equation and q are written as

$$a_n = 4(E_n - 2z_n J\rho_0^2(1+\langle\cos\phi\rangle^2))\frac{2m\rho_0^2}{\hbar^2} = \frac{E_n - B(1+\langle\cos\phi\rangle^2)}{A},$$
$$q = -8z_n\rho_0^2 J\langle\cos\phi\rangle\frac{2m\rho_0^2}{\hbar^2} = -\frac{B}{A}\langle\cos\phi\rangle,$$

where we define $A = \frac{\hbar^2}{8m\rho_0^2}$ and $B = 2z_n J\rho_0^2$. The eigenvalues are explicitly written as

$$E_n = Aa_n + B(1+\langle\cos\phi\rangle^2). \tag{26}$$

The allowed eigenfunctions are determined by the condition that the wave functions be single-valued, i.e., that $\psi_n(\phi + 2\pi) = \psi_n(\phi)$, or equivalently, that $y_n(v+\pi) = y_n(v)$. The allowed three lowest solutions, up to the order of q^2, are [25]

$$y_0(v,q) = \frac{1}{\sqrt{\pi}}\left[1 - \frac{q}{2}\cos 2v + q^2\left(\frac{\cos 4v}{32} - \frac{1}{16}\right)\right], \quad a_0 = -\frac{q^2}{2},$$

$$y_2(v,q) = \frac{2}{\sqrt{\pi}}\left[\cos 2v - q\left(\frac{\cos 4v}{12} - \frac{1}{4}\right) + q^2\left(\frac{\cos 6v}{384} - \frac{19\cos 2v}{288}\right)\right],$$
$$a_2 = 4 + \frac{5q^2}{12},$$

$$y_{-2}(v,q) = \frac{2}{\sqrt{\pi}}\left[\sin 2v - q\frac{\sin 4v}{12} + q^2\left(\frac{\sin 6v}{384} - \frac{\sin 2v}{288}\right)\right],$$
$$a_{-2} = 4 - \frac{q^2}{12},$$

where these are normalized like $\int_0^{2\pi}\psi_n(\phi)d\phi = 1$. Thus, the matrix elements for $\langle\cos\phi\rangle$ on the corresponding bases, $n = 0,2$, and -2, are

$$\langle\cos\phi\rangle = \begin{pmatrix} -\frac{q}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{5q}{12} & 0 \\ 0 & 0 & -\frac{q}{12} \end{pmatrix}. \tag{27}$$

From the mapping, we can get the self-consistent condition for the critical temperature of the phase ordering in terms of the parameters of the GL equation for sufficient or infinite long wires Eq. (23), which corresponds to the only consideration of the ground state ($n = 0$) in the quantum mechanics problem, and it takes the following form.

$$\langle\cos\phi\rangle = -\frac{q}{2}. \tag{28}$$

The transition temperature for phase ordering can be calculated by finding the temperature, where $\langle\cos\phi\rangle$ becomes zero. Here it is convenient to introduce the quantity $K = J\xi_0^2$. Thus, because with $A \to \frac{m^*\xi_0^4}{8\hbar^2\beta^2\Delta^2(t)}$ and $B \to \frac{2z_n K\Delta^2(t)}{\xi_0^2}$, we have

$$\frac{B}{A} \to \frac{16z_n\hbar^2\beta^2K\widetilde{\Delta}^4(t)}{m^*\xi_0^6} = 2\alpha\frac{g^4(t)}{t^2}. \tag{29}$$

It is convenient to define a variable $\kappa = \frac{8z_n\hbar^2\widetilde{\Delta}^4(0)K}{m^*\xi_0^6(k_BT_c^0)^2}$. In terms of κ, the condition determining the phase-ordering transition temperature becomes

$$t_c = \sqrt{\kappa}g^2(t_c). \tag{30}$$

Therefore, using $g(t) = \sqrt{1-t}$, this critical temperature becomes

$$t_c = \frac{\sqrt{\kappa}}{1+\sqrt{\kappa}}. \tag{31}$$

The temperature dependence of the order parameter obtained by minimalising the energy, Eq. (26) for infinitely long wires is shown in Fig. 1. This figure clearly shows that, within this phase-only mean-field approximation, there is a second order phase transition because the order parameter goes continuously to zero at the critical point. As expected, the critical temperature of the entire collection of wires is lower than the mean-field critical temperature of a single wire.

On the other hand, for finite length wires, contributions from excited states in the quantum mechanics problem need to be considered because the effective temperature is not zero. Including up to the order $|n| \leqslant 2$ for the solution of Mathieu's equation and using Eq. (22) the following self-consistent condition can be obtained,

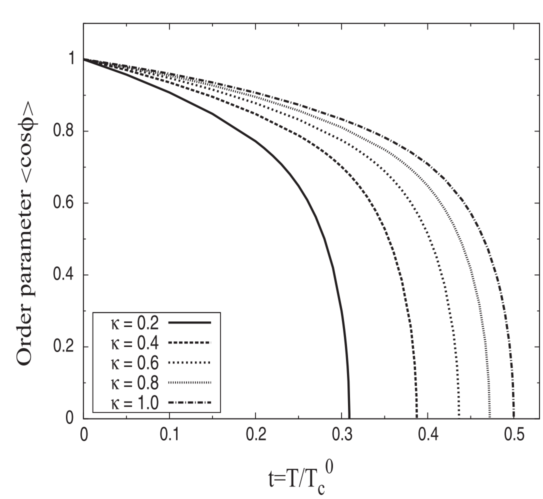

Fig. 1. Temperature-dependence of the order parameter for phase ordering, assuming infinitely long wires, as calculated in the mean-field approximation as described in the text, for $\kappa = 0.2, 0.4, 0.6, 0.8$, and 1.0 and no magnetic field.

$$-\frac{A}{B}q = \frac{-\frac{q}{2}e^{-\beta_{eff}E_0} + \frac{5q}{12}e^{-\beta_{eff}E_2} - \frac{q}{12}e^{-\beta_{eff}E_{-2}}}{e^{-\beta_{eff}E_0} + e^{-\beta_{eff}E_2} + e^{-\beta_{eff}E_{-2}}}, \quad (32)$$

where $\beta_{eff}E_n = \beta_{eff}(Aa_n + B(1 + \langle\cos\phi\rangle^2))$.

Therefore, we can get

$$\left(\frac{t}{1-t}\right)^2 = \kappa\frac{1 - \frac{2}{3}e^{-4x\frac{t}{1-t}}}{1 + 2e^{-4x\frac{t}{1-t}}}, \quad (33)$$

where we introduce $x = \eta z_{max}/\xi_0$, $\eta = [m^*\xi_0^2 k_B T_c^0/(8\hbar^2)](\xi_0^2/\widetilde{\Delta}^2(0)$, and we use the following mapping:

$$\beta_{eff}A \rightarrow \frac{m^*\xi_0^4}{8\hbar^2\beta\widetilde{\Delta}^2(t)}\frac{z_{max}}{\xi_0} = \frac{m^*\xi_0^2 k_B T_c^0}{8\hbar^2}\frac{\xi_0^2}{\widetilde{\Delta}^2(0)}\frac{z_{max}}{\xi_0}\frac{t}{1-t}$$
$$= \eta\frac{z_{max}}{\xi_0}\frac{t}{1-t}, \quad (34)$$

where $\eta = \frac{m^*\xi_0^2 k_B T_c^0}{8\hbar^2}\frac{\xi_0^2}{\widetilde{\Delta}^2(0)}$. Using the numerical values according to Tang et al. [1], $\eta \approx 1.4 \times 10^{-4}$. A plot of T_c versus κ for several lengths ($100\xi_0$, $1000\xi_0$, $2000\xi_0$, and $5000\xi_0$) and infinite length are given in Fig. 2. This figure shows that as the length of the wires increases, the phase-ordering critical temperature also increases.

3.2. Perpendicular magnetic field

The critical temperature for the presence of a vector potential parallel to the wires can be obtained by solving the non-Hermitian Eq. (21). Using $\psi_n(\phi) = e^{pv}F(v)$ with $v = \phi/2$ and $p = i\frac{2e^*\rho_0^2 B_{eff}}{ch}$, again this equation reduces to the standard Mathieu equation:

$$\frac{d^2F(v)}{dv^2} - (2q\cos 2v)F(v) = -a_v F(v), \quad (35)$$

where

$$a_v - p^2 = 4(E_n - 2z_n J\rho_0^2(1 + \langle\cos\phi\rangle^2))\frac{2m\rho_0^2}{\hbar^2}$$
$$= \frac{E_n - B(1 + \langle\cos\phi\rangle^2)}{A}, \quad (36)$$

and

$$q = -8z_n\rho_0^2 J\langle\cos\phi\rangle\frac{2m\rho_0^2}{\hbar^2} = -\frac{B}{A}\langle\cos\phi\rangle. \quad (37)$$

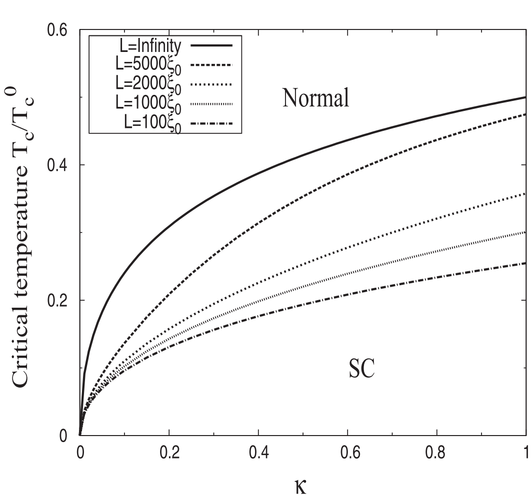

Fig. 2. Scaled critical temperature $t_c = T_c/T_c^0$ as a function of κ for several values of length of the wires, $100\xi_0$, $1000\xi_0$, $2000\xi_0$, $5000\xi_0$, and ∞, assuming $\xi_0 = 42$ Å.

The allowed eigenvalues are determined by the boundary condition that $\psi_n(\phi + 2\pi) = \psi_n(\phi)$, or equivalently $F(v + \pi) = \exp(-p\pi)F(v)$. Thus we are interested only in the Floquet solutions of the Mathieu equation with Floquet exponent $v = 2n + ip$, where $n = 0, \pm1, \pm2, \ldots$ These solutions are explicitly written as [25]

$$F_v(v) = c_0 e^{ivv}\left[1 - q\left(\frac{e^{2iv}}{4(v+1)} - \frac{e^{-2iv}}{4(v-1)}\right)\right], \quad (38)$$

where c_0 is a normalization constant. The eigenvalues are,

$$a_v = v^2 + \frac{q^2}{2(v^2-1)}. \quad (39)$$

The three lowest allowed solutions, up to the order of q^2, are [25]

$$\psi_{ip}(v) = \sqrt{\frac{1}{\pi}}\left(1 - q\frac{\cos 2v + p\sin 2v}{2(1+p^2)}\right), \quad a_{ip} = -\frac{q^2}{2(1+p^2)},$$
$$\psi_{2+ip}(v) = \sqrt{\frac{1}{\pi}}\left(e^{2iv} - \frac{q}{4}\left(\frac{e^{4iv}}{3+ip} - \frac{1}{1+ip}\right)\right), \quad a_{2+ip} = 4(1+ip) + \frac{q^2}{2(-p^2+4ip+3)},$$
$$\psi_{-2+ip}(v) = \sqrt{\frac{1}{\pi}}\left(e^{-2iv} + \frac{q}{4}\left(\frac{e^{-4iv}}{ip-3} - \frac{1}{ip-1}\right)\right), \quad a_{-2+ip} = 4(1-ip) + \frac{q^2}{2(3-p^2-4ip)}.$$

Left wave functions can be obtained from right wave function using $\psi_n^L(v, p) = \psi_n(v, -p)^*$.

The matrix elements for $\langle\cos\phi\rangle$ corresponding to $n = 0$, 2, and -2 are, using $q = -\frac{B}{A}\langle\cos\phi\rangle$,

$$\langle\cos\phi\rangle = \frac{1}{2}\begin{pmatrix} -\frac{q}{1+p^2} & 1 & 1 \\ 1 & \frac{q}{3+4ip-p^2} & \frac{1}{2(1+p^2)} \\ 1 & \frac{1}{2(1+p^2)} & \frac{q}{3-4ip-p^2} \end{pmatrix}. \quad (40)$$

The self-consistent condition for long wires takes the following form:

$$\langle\cos\phi\rangle = -\frac{q}{2(1+p^2)}. \quad (41)$$

Again, we can determine the transition temperature for phase ordering by finding where $\langle\cos\phi\rangle$ becomes zero. This condition with Eq. (29) becomes

$$t_c = \sqrt{\frac{\kappa}{1+p^2(t_c)}}g^2(t_c). \quad (42)$$

If we again use the approximation $g(t) = \sqrt{1-t}$, we can get

$$t_c = \frac{\sqrt{\kappa - f^2}}{1 + \sqrt{\kappa - f^2}}, \quad (43)$$

where we define $f = \frac{8\pi\hbar^2}{k_B T_c^0 m^* \xi_0^2}\frac{\widetilde{\Delta}^2(0)}{\xi_0^2}\frac{A_z\xi_0}{\Phi_0}$, and $\Phi_0 = hc/e^*$. The ordering condition may be written

$$p = i\frac{2e^*\rho_0^2 B_{eff}}{ch} \rightarrow \frac{8\pi\hbar^2}{k_B T_c^0 m^*\xi_0^2}\frac{\widetilde{\Delta}^2(0)}{\xi_0^2}\frac{A_z\xi_0}{\Phi_0}\frac{g^2(t)}{t} = f\frac{g^2(t)}{t}, \quad (44)$$

where $f = \frac{8\pi\hbar^2}{k_B T_c^0 m^* \xi_0^2}\frac{\widetilde{\Delta}^2(0)}{\xi_0^2}\frac{A_z\xi_0}{\Phi_0}$ and $\Phi_0 = hc/e^*$. When $f = 0$, this solution corresponds to the previous case (of zero magnetic field). A plot of T_c versus κ for $f = 0$, 0.2, 0.4, 0.6, and 0.8 is given in Fig. 3. This figure shows that for each $f(\neq 0)$, there is the critical interaction strength between wires, below which there is no phase ordering.

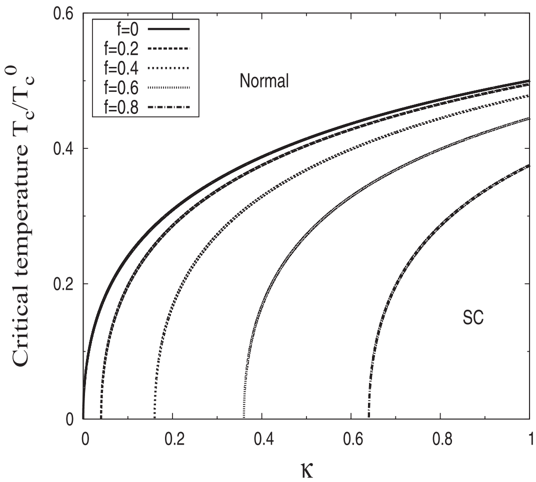

Fig. 3. Plot of the mean-field critical temperature for phase ordering, $t_c = T_c/T_c^0$, as a function of κ for several values of magnetic field strength and infinite length wires. The magnetic field strength is described by the parameter f, which takes on the values $f = 0.2$, $f = 0.4$, $f = 0.6$, and $f = 0.8$.

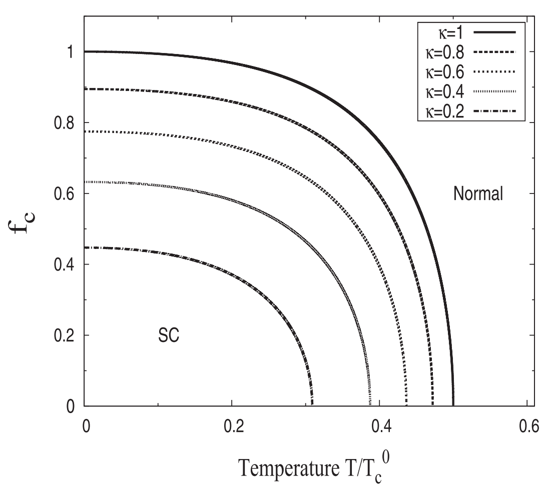

Fig. 4. Temperature dependence of the critical f_c of the vector potential strength parameter, for $\kappa = 0.2, 0.4, 0.6, 0.8$, and 1.0 and infinitely long wires, as computed in the mean-field approximation.

From Eq. (43), the condition of this critical value for κ is obtained by $\kappa \geqslant f^2$. Substituting corresponding values, for the above condition, we can get

$$z_n K \geqslant \frac{8\pi^2\hbar^2}{m^*\xi_0^2}\left(\frac{A_z\xi_0}{\Phi_0}\right)^2. \tag{45}$$

The critical f_c, above which phase ordering is broken, can be obtained from the equation

$$f_c = \frac{\sqrt{\kappa(1-t)^2 - t^2}}{1-t}. \tag{46}$$

Near the critical temperature for phase ordering, using the notation $\delta t = t_c - t = \frac{\sqrt{\kappa}}{1+\sqrt{\kappa}} - t$, this can be written

$$f_c \approx \sqrt{2}(1+\sqrt{\kappa})\kappa^{1/4}\sqrt{\delta t}. \tag{47}$$

In Fig. 4, we show this f_c for $\kappa = 0.2, 0.4, 0.6, 0.8$ and *1* as a function of a temperature.

4. Discussion

In this paper, we have presented a mapping between a Ginzburg–Landau free energy describing a collection of parallel one-dimensional superconducting wires in the presence of a vector potential along the wires and a two-dimensional quantum mechanical problem describing a collection of particles in the presence of a perpendicular imaginary magnetic field. Moreover, in the case of weak links between wires, we have obtained, using a mean-field approximation, the phase diagrams for the system both in the presence and the absence of this vector potential.

Next, we discuss the parameters used in this paper. In our calculations, we have used the numerical values of the various parameters appropriate to those of a single-walled carbon nanotube, which according to Tang et al. [1], is superconducting, with a relatively high transition temperature $T_c^0 = 15$ K or $k_B T_c^0 = 1.3$ meV. The corresponding values of the other parameters are $\alpha_0 T_c^0 = 6$ meV, $\gamma = 1.3$ meV Å, $m^* = 0.36$ me, and $\xi_0 = \frac{\hbar}{\sqrt{2m^*\alpha_0 T_c^0}} = 42$ Å. Using these values, we can obtain the following values for κ and f : $\kappa = \frac{z_n K}{8.6\times10^{-6}\ \text{meV}}$ and $f = 1.7\times10^4 \frac{A_z z_{max}}{\Phi_0}\frac{\xi_0}{z_{max}}$.

Next, we discuss our use of the GL free energy functional. In principle, this free energy functional is applicable only near the critical temperature, $T - T_c^0 \ll T_c^0$. Thus, except for temperatures near the critical temperature T_c^0, the predictions of this functional may not be accurate, although they could be improved by including terms beyond fourth order in the order parameter in the Ginzburg–Landau equation.

We want to comment the effect on the interaction term by a magnetic field. When there is a magnetic field, the phase difference needs to be replaced by $\phi_i - \phi_{i+1} - \frac{2\pi}{\Phi_0}\int \vec{A}\cdot d\vec{l}$, where the integration is between different wires. However, because the direction of vector potential is taken in the direction of the wires, z, there is no contribution from the integral on the phase difference.

In this paper, for simplification we only consider the periodic boundary condition, where the wires are considered as circular loops, or enough long straight wires to ignore the effect of the boundary. When wires are sufficient long, the effect of the boundary conditions may not change the physical properties of the system when there is no vector potential. However, these boundary conditions may affect the properties of the system drastically when $\vec{A} \neq 0$. If the length of the wire is z, then the area of the loop is $\pi[z/(2\pi)]^2 = z^2/(4\pi)$. If the flux through the loop is Φ, then the average field in the loop is $B_{av} = 4\pi\Phi/z^2$. But we also have $zA_z = \Phi$ from $\nabla \times A = \vec{B}$, so $A_z = zB_{av}/(4\pi)$. Thus, as the length of the wires z gets very large, A_z would also get very large, for fixed B_{av} (or fixed flux through the loop). For straight wires, A_z remains fixed as the length of the wire becomes very large. Thus, the effect of the boundary condition could change the physics of the system considerably.

If a vector potential A_z is constant along the wires and is the same along each wire like this paper, a magnetic field with this condition would give a one-dimensional array of large wires or loops (like a solenoid) but not applicable to a 2D array of wires. For example, if vector $\vec{A} = (0, 0, -Bx)$, which gives B field in the y -direction, is considered, 1D array of wires can be arranged in the y-direction but for a magnetic field perpendicular to a 2D array of wires, the A_z's on different wires would not all be equal. However, our mapping is applicable in 2D arrays if we accept that A_z does not have to be the same along each wire. If we eliminate the condition that A_z is constant and is not a function of z, 2D array of wires may be considered although the corresponding quantum problem may become time dependent or have more complex potential.

Moreover, our theory neglects the effects of disorder, which plays an important role on balk superconductors. With these degrees of freedom, the properties of the system may be changed. Thus, it might be an interest to consider these cases for our future research.

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