Model for a macroscopically disordered conductor with an exactly linear high-field magnetoresistance

V. Guttal and D. Stroud
Department of Physics, The Ohio State University, Columbus, Ohio 43210-1106, USA
(Received 21 December 2004; published 17 May 2005)

We calculate the effective resistivity of a macroscopically disordered two-dimensional conductor consisting of two components in a perpendicular magnetic field. When the two components have equal area fractions, we use a duality theorem to show that the magnetoresistance is nonsaturating and at high fields varies exactly linearly with the magnetic field. At other compositions, an effective-medium calculation leads to a saturating magnetoresistance. We briefly discuss possible connections between these results and magnetoresistance measurements on heavily disordered chalcogenide semiconductors.

DOI: 10.1103/PhysRevB.71.201304 PACS number(s): 75.47.De, 61.43.Hv, 72.15.Gd

The resistivity of most homogeneous materials (metals or semiconductors) increases quadratically with magnetic field $H$ at low fields, and generally saturates at sufficiently large $H$. Exceptions may occur for materials with Fermi surfaces allowing open orbits, or for compensated homogeneous semiconductors, where the resistivity may increase without saturation, usually proportional to $H^2$. Under some special conditions, the magnetoresistance can be linear in magnetic field.

Recently, a remarkably large transverse magnetoresistance (TMR) has been observed in the doped silver chalcogenides Ag$_{2+x}$Se and Ag$_{2+y}$Te. In these materials, over the temperature range from 4 to 300 K, the resistivity increases approximately linearly with $H$ up to fields, applied perpendicular to the direction of current flow, as high as 60 T. Moreover, the TMR is especially large and most clearly linear at pressures where the Hall resistivity changes sign. Because of this linearity, these materials may be useful as magnetic field sensors even at megagauss fields.

But beyond the possible applications, the origin of the effect remains mysterious. According to conventional theories, such narrow gap semiconductors should have a saturating TMR. Furthermore, since these materials contain no magnetic moments, a spin-mediated mechanism seems unlikely.

There are presently two proposed explanations for this quasi-linear TMR. The first is a quantum theory of magnetoresistance (MR). The second proposed mechanism is that this nonsaturating TMR arises from macroscopic sample inhomogeneities. Such inhomogeneities could produce large spatial fluctuations in the conductivity tensor and hence a large TMR, especially at large $H$. This explanation seems plausible because the chalcogenides probably have a granular microstructure, and hence a spatially varying conductivity.

The effective conductivity of a macroscopically disordered conductor with an exactly linear high-field magnetoresistance is given by $\sigma(x) = \frac{1}{2} \left( \sigma_A + \sigma_B \right) + \frac{1}{2} \left( \sigma_A - \sigma_B \right) \cos(2\pi x / L)$, where $\sigma_A$ and $\sigma_B$ are the conductivities of the two components. The effective conductivity $\sigma(x)$ is related to the MR by $R_{MR} = \frac{\sigma(x)}{\sigma}$. The effective conductivity tensor can be calculated using a duality theorem, which relates the MR to the conductivity tensor. The MR is given by $R_{MR} = \frac{\sigma(x)}{\sigma} = \frac{1}{2} \left( \frac{\sigma_A - \sigma_B}{\sigma} \right)$, where $\sigma_A$ and $\sigma_B$ are the conductivities of the two components. The effective conductivity tensor can be calculated using a duality theorem, which relates the MR to the conductivity tensor. The MR is given by $R_{MR} = \frac{\sigma(x)}{\sigma} = \frac{1}{2} \left( \frac{\sigma_A - \sigma_B}{\sigma} \right)$. The effective conductivity tensor can be calculated using a duality theorem, which relates the MR to the conductivity tensor. The MR is given by $R_{MR} = \frac{\sigma(x)}{\sigma} = \frac{1}{2} \left( \frac{\sigma_A - \sigma_B}{\sigma} \right)$. The effective conductivity tensor can be calculated using a duality theorem, which relates the MR to the conductivity tensor. The MR is given by $R_{MR} = \frac{\sigma(x)}{\sigma} = \frac{1}{2} \left( \frac{\sigma_A - \sigma_B}{\sigma} \right)$. The effective conductivity tensor can be calculated using a duality theorem, which relates the MR to the conductivity tensor. The MR is given by $R_{MR} = \frac{\sigma(x)}{\sigma} = \frac{1}{2} \left( \frac{\sigma_A - \sigma_B}{\sigma} \right)$.
experiments\(^6\) (which are, however, carried out for 3D samples; see below). If the carriers have the same sign, no exact statements are possible, even at \(p_A=1/2\). But even in this case the EMA predicts a linear TMR precisely at \(p_A=1/2\), though smaller than for carriers of opposite sign.

We first prove the exact linearity of the TMR at \(p_A=1/2\) for carriers of opposite sign and opposite mobility, using a duality argument. We consider a two-dimensional (2D) conductor with a spatially varying conductivity tensor \(\sigma(x)\), and denote the effective conductivity tensor by \(\sigma_e\). \(\sigma_e\) is a 2 \(\times\) 2 tensor defined by \((J)=\sigma_e(E)\), where \(J\) and \(E\) are the position-dependent current density and electric field, and \((\cdot,\cdot)\) denotes a spatial average in the limit of a large sample and suitable boundary conditions (as discussed, for example, in Ref. 15). \(\sigma_e\) is the quantity that would be measured as the sample conductivity in an experiment. To calculate \(\sigma_e\), we use a duality theorem,\(^{10}\) which states that

\[
\sigma_e[\sigma(x)]\sigma_e[\sigma^{-1}(x)]=I, \tag{1}
\]

where \(I\) is the 2 \(\times\) 2 unit matrix. Here \(\sigma_e[\sigma(x)]\) denotes the effective conductivity tensor of a material whose local conductivity tensor is position-dependent and equal to \(\sigma(x)\).

Thus, the product of \(\sigma_e\) for the system of interest, and that of a hypothetical “dual composite” whose local conductivity tensor \(\sigma_d(x)\) is the local resistivity tensor of the original material, equals the unit tensor.

We now apply this theorem to the following special case. Let the two components each have a free-electron conductivity, but carriers of opposite signs. For the first component

\[
\sigma_{A,xx} = \sigma_{A,yy} = \frac{\sigma_{A,0}}{1 + H^2}, \tag{2}
\]

\[
\sigma_{A,xy} = -\sigma_{A,yx} = \frac{\sigma_{A,0}H}{1 + H^2}, \tag{3}
\]

where \(\sigma_{A,0}\) is the zero-field conductivity. The dimensionless magnetic field \(H=\mu_B B/c\), where \(\mu_B=e\tau_m/m_A\) is an effective mobility of carriers of type \(A\), \(m_A\) is their effective mass, \(e>0\) is the electron charge magnitude, and \(\tau_m\) a characteristic relaxation time. For the second component, we assume

\[
\sigma_{B,xx} = \sigma_{B,yy} = \frac{\sigma_{B,0}}{1 + k^2H^2}, \tag{4}
\]

\[
\sigma_{B,xy} = -\sigma_{B,yx} = \frac{\sigma_{B,0}kH}{1 + k^2H^2}, \tag{5}
\]

with the dimensionless constant \(k=-1\) (i.e., the two types of charge carriers have opposite signs). We also introduce \(\mu_B=k\mu_A\) as the effective mobility of type-B carriers. Finally, we assume that the composite contains an areal fraction \(p_A=1/2\) \((i=A\ or\ B)\) of each component, and that the geometry is symmetric. “Symmetric” means that, if the components \(A\) and \(B\) were interchanged, \(\sigma_e\) of the film will remain the same in the thermodynamic limit. There are many geometries, both ordered (e.g., checkerboard) and random, which are symmetric by this definition. If we make the usual Drude assumption that \(\sigma_{i,0}=n_i e^3/\mu_i\) \((i=A,B)\), where \(n_i\) is the number density of carriers of type \(i\), then Eqs. (2)–(5) imply (i) that there are equal areal fractions of positive and negative charge carriers (but not that the total numbers of positive and negative charge carriers are equal); and (ii) that the mobilities \(\mu_A\) and \(\mu_B\) are equal and opposite, so that \(\langle\mu\rangle=2\mu_A B p = 0\).

Given these assumptions, the tensors \(\sigma_A\) and \(\sigma_B\) satisfy the remarkable relationship

\[
\sigma_A^{-1} = \frac{1 + H^2}{\sigma_0^2} \sigma_B, \tag{6}
\]

where \(\sigma_0=\sqrt{|\sigma_{A,0}\sigma_{B,0}|}\). Since we have an equal proportion of components \(A\) and \(B\), distributed in some symmetrical (and isotropic) fashion, the dual composite has a conductivity tensor

\[
\sigma_d(x) = \frac{1 + H^2}{\sigma_0^2} \sigma_e[\sigma(x)], \tag{7}
\]

where \(\sigma_e(x)\) means the conductivity of a composite in which the \(A\) and \(B\) components are interchanged. Since \(\sigma_d\) is just a multiple of the original conductivity tensor \(\sigma(x)\), but with \(A\) and \(B\) components interchanged, and since by the assumption of a symmetric composite \(\sigma_e[\sigma(x)]=\sigma_e[\sigma_d(x)]\), it follows that

\[
\sigma_e[\sigma_d(x)] = \frac{1 + H^2}{\sigma_0^2} \sigma_e[\sigma(x)]. \tag{8}
\]

We now apply Eq. (1) to this model, with the result

\[
\frac{1 + H^2}{\sigma_0^2} \sigma_e[\sigma(x)] = I. \tag{9}
\]

A physically acceptable solution to Eq. (9) must have the diagonal elements of \(\sigma_e\) equal and positive, and off-diagonal elements equal and opposite. It is readily shown algebraically that the only such solution is

\[
\sigma_e[\sigma(x)] = \frac{1}{\sqrt{1 + H^2}} \sigma_d I. \tag{10}
\]

The corresponding resistivity tensor \(\rho_e\) is

\[
\rho_e = \sigma_0^{-1} \sqrt{1 + H^2} I. \tag{11}
\]

The TMR is defined by the relation \(\Delta \rho_{e,xx}(H)=\rho_{e,xx}(H) - \rho_{e,xx}(0)/\rho_{e,xx}(0)\). For this model, \(\Delta \rho_{e,xx}(H)=\sqrt{1 + H^2} - 1\) becomes linear in \(H\) for large enough \(H\), and the corresponding Hall coefficient \(R_H=\rho_{e,xy}(H)/H=0\). Thus, this calculation appears to reproduce the numerical results of Ref. 8, but analytically.

Since the duality argument is not sufficient to determine \(\sigma_e\) for \(p_A \neq 1/2\), we have used the EMA for such concentrations. The EMA is a simple mean-field approximation in which the local electric fields and currents are calculated as if a given region is surrounded by a suitably averaged environment. For the present model the EMA becomes\(^{22}\)
At all other concentrations, we also assume that the two components have tested it for easily solved numerically. Evidently, and as can be shown explicitly from the EMA distributed in compact, approximately circular regions. Then the components of the resistivity tensor, the diagonal elements are given by Eqs. (2)–(5), with $\sigma_{i,0} = 2$. The mobilities of the two carriers are assumed to have the same magnitudes: $|\mu_A| = |\mu_B|$. We assume that $\sigma_A$ and $\sigma_B$ satisfy $\sigma_{xxxx} = \sigma_{yyyy}$; $\sigma_{yx} = -\sigma_{xy}$. Then the components of $\sigma_{i}$ satisfy $\sigma_{i,xx} = \sigma_{i,yy}$; $\sigma_{i,xy} = -\sigma_{i,yx}$. We also assume that the two components $A$ and $B$ are distributed in compact, approximately circular regions. Then $\Gamma = -I/(2\sigma_{e,xy})$. 22 With these assumptions, Eq. (12) reduces to two coupled algebraic equations for $\sigma_{e,xx}$ and $\sigma_{e,xy}$ which are easily solved numerically.

To illustrate the EMA predictions for $p_A \neq \frac{1}{2}$, we calculate $\sigma_e$ for $\sigma$ given by Eqs. (2)–(5). The resulting elements of the resistivity tensor, $\rho_{e,xx} = \sigma_{e,xx}/(\sigma_{e,xx}^2 + \sigma_{e,xy}^2)$, $\rho_{e,xy} = -\sigma_{e,xy}/(\sigma_{e,xx}^2 + \sigma_{e,xy}^2)$, are plotted in Fig. 1 for $\sigma_{0,0}/\sigma_{0,0} = 2$. Evidently, and as can be shown explicitly from the EMA equations, $\rho_{e,xx}$ is strictly linear in $H$ only at $p_A = 1/2$. At all other concentrations, $\rho_{e,xx}(H)$ saturates (i.e., approaches a constant) at large $H$, but at a value much larger than $\rho_{e,xx}(H=0)$. It is easily shown that the saturation value of $\Delta \rho_{e,xx}(p_A) \approx \lim_{H \to \infty} \rho_{e,xx}(H, p_A)/\rho_{e,xx}(0, p_A) - 1 \propto p_A$ on both sides of the percolation threshold $p_c = 1/2$. Figure 1 also shows that the effective Hall resistivity $\rho_{e,xy}$ changes sign just at the concentration where $\rho_{e,xx}$ varies asymptotically linearly with $H$.

We have also solved the EMA for a composite described by Eqs. (2)–(5) but for the more general case in which $k \neq -1$. Then $k > 0$ and $k < 0$ correspond, respectively, to carriers with mobilities of the same and opposite signs.

In Fig. 2 we show the EMA results for this model. Specifically, we show $\rho_{e,xx}(H, p_A)$ and $\rho_{e,xy}(H, p_A)$ with $p_A = 1/2$, $\sigma_{A_0} = \sigma_{B_0}$, and several choices of $k$ corresponding to carriers of both opposite and the same sign. The case $k = 1$ actually corresponds to a homogeneous free-electron metal. For all other values of $k$, the TMR is asymptotically linear, the linear behavior is evident even at moderate fields ($H \sim 1$). However, the linear slope is larger when the carriers have opposite signs. We emphasize that these results are obtained in the EMA. The duality arguments do not give any predictions for $\rho_{e,xx}$ except when the carriers have opposite signs and opposite mobilities.

In Figs. 3 and 4, we plot the resistivity $\rho_{e,xx}$ and Hall coefficient $R_H = \rho_{e,xy}/H$ as a function of $p_A$ for $H = 1$ and $H = 10$. In both cases, we assume that $\sigma_{A_0} = \sigma_{B_0}$ and $|\mu_A| = |\mu_B|$. $\rho_{e,xx}$ has a peak at $p_A = 1/2$, which sharpens, as a function of $p_A$, as $H$ increases. Similarly, the Hall coefficient $R_H$ changes sign at $p_A = 1/2$, and the change occurs over a narrower and narrower regime of $p_A$ as $H$ increases.

The present results agree qualitatively with the experiments of Lee et al.23 which also show that the TMR peaks at pressures where the Hall coefficient changes sign. But this agreement should be viewed cautiously. In particular, the measurements of Ref. 6 are carried out on a 3D sample, while our calculations are for a 2D system. The present work would also apply to a 3D system with a columnar microstructure—that is, a system in which the conductivity tensor $\sigma(x)$ is independent of the third dimension, $z$ —and the applied field $B \parallel z$, but the samples of Ref. 6 if inhomogeneous, are most likely composed of small compact grains.

We have calculated $\sigma_e$ for a 3D granular sample with carriers of opposite signs, using the EMA, and find results similar to those shown here for 2D samples. These 3D calculations will be presented elsewhere.

The TMR of the present model is very large—$\Delta \rho_{e,xx}(H, 1/2) \sim 10$ for $H \approx 10$—and remains approximately linear down to fields as low as $H \sim 1$–2. By contrast, other models of TMR that arises from inhomogeneities produce only a small TMR, or, if a large TMR, $\Delta \rho_{e,xx}(H)$ does not vary linearly with $H$.15,16
In summary, we have presented a simple model of a 2D macroscopically inhomogeneous material, whose TMR is asymptotically linear in magnetic field, and whose corresponding Hall coefficient vanishes. The model has several unusual properties that make it likely to be realized only in special circumstances. First, Eqs. (2)–(5) imply that the carriers have equal and opposite mobilities \( m_A = -m_B \). Secondly, the linearity occurs only if the composite has a symmetric geometry at \( p_A = 1/2 \). But given these features, the TMR, arising from a perpendicular to the sample, is asymptotically exactly linear in \( B \). This is an analytically soluble model for TMR due to macroscopic inhomogeneities, which produces a linear TMR at high concentrations of inhomogeneities.

This work was supported by the National Science Foundation, through Grant No. DMR04-13395. We also benefited from the facilities of the Ohio Supercomputer Center.

1 For a discussion, see, e. g., N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders, Fort Worth, 1975), Chap. 12.
3 For a recent short review, see, e. g., A. A. Abrikosov, Europhys. Lett. 49, 789 (2000), and references therein.
23 V. Guttl and D. Stroud (unpublished).