Two Ways of Getting B-Field of Magnetic Dipole

As I implied in class, there are two different ways of getting the magnetic induction $\mathbf{B}$ for a magnetic dipole $\mathbf{m}$. The first is that specifically discussed in Jackson. It involves writing

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1}$$

and using the result (for a magnetic dipole moment at the origin)

$$A(\mathbf{x}) = \frac{\mu_o}{4\pi}\frac{\mathbf{m} \times \mathbf{x}}{r^3}, \tag{2}$$

where I have written

$$\mathbf{x} = r\hat{\mathbf{r}}. \tag{3}$$

If one carries out the curl operation, one finds that $\mathbf{B}(\mathbf{x})$ takes the form

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_o}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \hat{\mathbf{m}}) - \mathbf{m}}{r^3}. \tag{4}$$

This form for the magnetic induction is accurate so long as $r \gg a$, where a is a typical linear dimension of the magnetic dipole.

The magnetic induction $\mathbf{B}$ of a magnetic dipole can also be obtained from the *magnetic scalar potential.* Outside the dipole, there is no free current density, so $\nabla \times \mathbf{H} = 0$. Also, if this is a region of non-magnetic material, $\mathbf{B} = \mu_0\mathbf{H}$, and therefore $\nabla \cdot \mathbf{H} = \mu_0^{-1}\nabla \cdot \mathbf{B} = 0$. Therefore, $\mathbf{H}$ can be written as the gradient of a scalar potential,

$$\mathbf{H} = -\nabla\Phi \tag{5}$$

where

$$\nabla^2\Phi = 0. \tag{6}$$

The potential Φ should be azimuthally symmetric with respect to $\mathbf{m}$. Therefore, the leading term at large distances must have the form

$$\Phi = C\frac{\mathbf{m} \cdot \mathbf{x}}{r^3}, \tag{7}$$

where C is a constant to be determined. Therefore, $\mathbf{B}$ has the form

$$\mathbf{B} = -\mu_0 \nabla \Phi = \mu_0 C \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}}{r^3}. \tag{8}$$

Comparing eqs. (8) and (4), we see that they are consistent only if

$$C = \frac{1}{4\pi} \tag{9}$$

and therefore

$$\Phi = \frac{1}{4\pi} \frac{\mathbf{m} \cdot \mathbf{x}}{r^3}. \tag{10}$$

In summary, one can obtain the magnetic induction $\mathbf{B}$ of a magnetic dipole $\mathbf{m}$ either as $\mathbf{B} = \nabla \times \mathbf{A}$, with $\mathbf{A}$ given by eq. (2), or as $\mathbf{B} = -\mu_0 \nabla \Phi$, with Φ given by eq. (11).

Note that when one uses the scalar potential method, the scalar potential for $\mathbf{H}$ has the same form as that for the electric field $\mathbf{E}$ of an electric dipole, provided that one makes the replacements $\mathbf{p} \to \mathbf{m}$ and $1/(4\pi\epsilon_0) \to 1/(4\pi)$.