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New exact results for the Hall coefficient and magnetoresistance of inhomogeneous two-dimensional metals

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Several new theorems are proven for the magnetotransport coefficients of inhomogeneous normal-superconducting (N/S) and normal-insulating (N/I) films in a perpendicular magnetic field, under the assumption that macroscopic quantum effects such as Josephson coupling can be neglected. For N/S composites, at any field H, the Hall coefficient $R_H(p,H)\propto(p_c-p)^{2s}$ and the resistivity $\rho(p,H)\propto(p_c-p)^s$ where p is the volume fraction of superconductor and p_c is the percolation threshold. For N/I composites, as long as there is a connected metallic path across the sample, we show that both $R_H(p,H)$ and the ratio $\rho(p,H)/\rho(p,0)$ are independent of p and equal to the corresponding quantities for the normal metal.

I. INTRODUCTION

Thin metal films prove to have a wide variety of unusual transport properties in the presence of an applied perpendicular magnetic field. The most spectacular and fundamental of these is undoubtedly the quantum-Hall effect, which has by now acquired an extensive body of literature.[1] But there are other unusual properties displayed by thin superconducting films. For example, the Kosterlitz-Thouless vortex-unbinding transition sometimes reported in these films is strongly affected by an applied magnetic field.[2] Moreover, experiments on thin weak-coupled superconducting arrays show that these have resistivities and other transport properties which vary periodically with magnetic fields.[3] All of these properties, both for normal and superconducting films, are strongly influenced by impurities and other inhomogeneities. An important issue, therefore, is to explain the influence of such defects upon the transport properties of thin metallic films in a magnetic field.

The purpose of this Rapid Communication is to prove several *exact* results about the transport properties of inhomogeneous thin films in an applied perpendicular magnetic field. The results apply to *two-phase* films composed of regions of normal metal (N) and insulator (I), as well as to films with normal regions and superconducting (S) regions. The films are supposed to be sufficiently thin that all transport of interest occurs in the direction parallel to the film. They supplement what is, to our knowledge, the only previous exact result for transport in such inhomogeneous films,[4] which states that the low-field Hall coefficient of a N-I film above the percolation threshold for conduction is just that of the metal. While the kinds of inhomogeneities considered here are not those usually discussed in conventional transport theory, nevertheless, it is always valuable to have available any exact results, especially for such novel physical systems as those considered here. Our calculations are presented in that spirit.

The remainder of this paper is organized as follows. A necessary lemma is proven in Sec. II. The theorems of interest are then derived in Sec. III. Finally, in Sec. IV, a discussion is given of the potential relevance of our results to measurements on real materials.

II. LEMMA

We consider an inhomogeneous medium described by a conductivity tensor $\overleftrightarrow{\sigma}(\vec{x})$ which is a function of position $\vec{x}$. (The lemma to be proved here is valid for *both* two- and three-dimensional materials.) One way of defining the *effective* conductivity tensor $\overleftrightarrow{\sigma}_e$ of the inhomogeneous medium is the following. Let the medium be of volume V, bounded by surface S, and let the electrostatic potential $\Phi(\vec{x})$ on the surface be

$$\Phi(\vec{x})=-\vec{E}_0\cdot\vec{x}\text{ on } S\ . \tag{1}$$

The electrostatic equations are then $\vec{\nabla}\cdot\vec{J}=0$, $\vec{\nabla}\times\vec{E}=0$, and these, together with the constitutive relations,

$$\vec{J}(\vec{x})=\overleftrightarrow{\sigma}(\vec{x})\vec{E}(\vec{x})\ , \tag{2}$$

$$\vec{E}=-\vec{\nabla}\Phi\ , \tag{3}$$

lead to

$$\sum_{i=1,j=1}^{3}\partial_i\sigma_{ij}(\vec{x})\partial_j\Phi=0\ , \tag{4}$$

which is the tensorial analog of the usual Laplace's equation satisfied by the electrostatic potential.

The effective conductivity $\overleftrightarrow{\sigma}_e$ is now defined by the relation

$$\langle\vec{J}\rangle=\overleftrightarrow{\sigma}_e\langle\vec{E}\rangle\ , \tag{5}$$

where the triangular brackets denote volume (or area) averages,

$$\langle\vec{J}\rangle=\frac{1}{V}\int d\vec{x}\,\vec{J}(\vec{x})\ ,\qquad \langle\vec{E}\rangle=\frac{1}{V}\int d\vec{x}\,\vec{E}(\vec{x})\ . \tag{6}$$

For the given boundary conditions, it is readily shown that $\langle\vec{E}\rangle=\vec{E}_0$ whatever the inhomogeneous medium within V, so that determining $\vec{\sigma}_e$ reduces to the problem of finding $\langle\vec{J}\rangle$.

We now write $\vec{\sigma}(\vec{x})$ in the form

$$\vec{\sigma}(\vec{x})=\vec{\sigma}_s(\vec{x})+\vec{\sigma}_a(\vec{x}) \ , \tag{7}$$

where $\vec{\sigma}_s$ and $\vec{\sigma}_a$ are symmetric and antisymmetric tensors. This is a particularly convenient separation in the presence of an applied field, for according to the Onsager reciprocity relations, $\vec{\sigma}_s$ and $\vec{\sigma}_a$ are, respectively, even and odd functions of the components of the applied magnetic field $\vec{H}$. The lemma to be proved applies to the special case when $\vec{\sigma}_a$ is *position independent,* i.e., nonrandom. In that case we shall show that $\vec{\sigma}_e$ is simply given by

$$\vec{\sigma}_e=(\vec{\sigma}_e)_s+\vec{\sigma}_a \ , \tag{8}$$

where $(\vec{\sigma}_e)_s$ is the effective conductivity tensor for a medium of the same geometry but with $\vec{\sigma}_a=0$.

To prove the lemma, we first show that, if $\vec{\sigma}_a$ is position independent, then,

$$\vec{E}(\vec{x})=\vec{E}_s(x) \ , \tag{9}$$

where $\vec{E}_s(x)$ is just the same electric field that would be present if $\sigma_a=0$, but under the same boundary conditions. This follows directly from Eq. (4): if a constant antisymmetric tensor is added to $\vec{\sigma}_s(\vec{x})$ in Eq. (4), the partial differential equation is unchanged. Since Φ satisfies the same equation and boundary conditions as with $\vec{\sigma}_a=0$, it is identical to the case where $\vec{\sigma}_a=0$ from which Eq. (9) follows immediately.

From the definition (5) we have, for $\vec{\sigma}_a=\text{const}$,

$$\langle\vec{J}\rangle=\langle\vec{\sigma}_s\vec{E}\rangle+\vec{\sigma}_a\langle E\rangle \ . \tag{10}$$

But since $\vec{E}$ is everywhere the same as if $\vec{\sigma}_a=0$, it follows from Eq. (5) that the first term on the right is

$$\langle\vec{\sigma}\vec{E}\rangle=(\vec{\sigma}_e)_s\langle\vec{E}\rangle \ . \tag{11}$$

Equations (5), (10), and (11) then immediately imply the desired result: Eq. (8). In essence, a uniform *antisymmetric* part of the conductivity can be incorporated trivially into the general theory of the inhomogeneous medium, and this is the desired lemma.

III. SOME EXACT RESULTS FOR TWO-DIMENSIONAL CONDUCTORS

We now incorporate the result just proved into a model N/S composite. Let the composite be made up of a volume fraction $1-p$ of normal metal of conductivity $\vec{\sigma}_N=\vec{\sigma}_N^s+\vec{\sigma}_N^a$ where $\vec{\sigma}_N^s$ and $\vec{\sigma}_N^a$ are, respectively, the symmetric and antisymmetric parts of the tensor, and a volume fraction p of superconductor. We model the superconductor by the tensor

$$\vec{\sigma}_s=\lim_{\sigma\to\infty}[\sigma\vec{I}+\vec{\sigma}_N^a] \ , \tag{12}$$

where $\vec{I}$ is the 2×2 unit tensor. This seemingly artificial and unphysical addition of the antisymmetric term is included so as to permit the desired theorem to be proved below. Note that the extra term is totally harmless: a superconductor with the conductivity (12) still behaves like a classical perfect conductor, maintaining a zero electric field, for example; and so, writing the conductivity in this way wreaks no havoc with the essential physics of the composite.

If the magnetic field $\vec{H}$ is perpendicular to the film, then the symmetric and antisymmetric parts of the normal metal conductivity $\vec{\sigma}_N$ will obey[5]

$$\vec{\sigma}_N^s=a\vec{I} \ ; \qquad \vec{\sigma}_N^a=b\vec{K}; \ \ \vec{K}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \ . \tag{13}$$

Theorem (8) then implies that the *effective* conductivity of the composite satisfies

$$\vec{\sigma}_e=\sigma_e^s\vec{I}+b\vec{K} \ , \tag{14}$$

where σ_e^s is the effective conductivity of a composite made up of volume fraction p of superconductor with *infinite* conductivity and volume fraction $1-p$ of normal conductor with *scalar* conductivity a. Thus, we have related $\vec{\sigma}_e(\vec{H})$ *exactly* to $\vec{\sigma}_e(0)$—the essential result of this paper.

Usually one measures elements not of the conductivity but of the resistivity tensor $\vec{\rho}_e=\vec{\sigma}_e^{-1}$; from Eq. (14) this is given by

$$\vec{\rho}_e=[(\sigma_e^s)^2+(b^2)]^{-1}[\sigma_e^s\vec{I}-b\vec{K}] \ . \tag{15}$$

The most interesting behavior occurs near p_c, the volume fraction at which the superconductor first forms a connected path across the composite. If we assume, as is generally believed,[6] that σ_e^s diverges near p_c with a characteristic exponent, i.e., that

$$\sigma_e^s=Aa(\Delta p)^{-s} \ , \tag{16}$$

where $\Delta p=(p_c-p)/p_c$ and $s\sim1.1$–1.3 in two dimensions,[7] and A is a numerical factor of order 1,[7] then it follows that the transverse magnetoresistance $(\rho_e)_{xx}$ is given by

$$(\rho_e)_{xx}\sim\frac{aA(\Delta p)^{-s}}{[aA(\Delta p)^{-s}]^2+b^2} \ . \tag{17}$$

The critical behavior near p_c is thus

$$[\rho_e(p,H)]_{xx}\sim(\Delta p)^{+s} \ , \qquad R_H(p,H)\equiv(\rho_e)_{xy}/H\sim(\Delta p)^{+2s} \ . \tag{18}$$

If we make the additional assumption that $a\propto1/H^2$, $b\propto1/H$ at high fields, then we obtain the further result

$$\frac{[\rho_e(p,H=\infty)]_{xx}}{\rho_e(p,H=0)}\propto(\Delta p)^{-2s} \ , \tag{19}$$

which states that the magnetoresistance, somewhat surprisingly, saturates at an ever higher value as p approaches p_c from below. These results are *exact* under the given assumptions at any field strength, and should be readily testable in an appropriate experiment, provided ambiguities associated with proximity-effect coupling between S regions, i.e., with quantum effects, can be suppressed.

We turn next to the behavior of two-dimensional N/I composites near the percolation threshold. Our results here follow immediately from an elegant duality theorem proved by Mendelson.[8] For the case of interest, this theorem simply takes the form

$$\vec{\sigma}_e[\vec{\sigma}^{-1}(\vec{x})]\cdot\vec{\sigma}_e[\vec{\sigma}(\vec{x})]=\vec{I} \ , \tag{20}$$

where $\vec{\sigma}_e[\vec{\sigma}(\vec{x})]$ is the *effective* conductivity tensor of a two-dimensional composite in which the local conductivity is $\vec{\sigma}(\vec{x})$. [Equation (21) is extracted from Mendelson's theorem for the special circumstance that $\vec{\sigma}(\vec{x})$ is invariant under a rotation about the z axis, as is the case here.] The dual of the N/S composite is a two-phase medium with a volume fraction p of insulator (of zero conductivity) and $1-p$ of conductor of conductivity

$$\vec{\sigma}_N^d = \vec{\sigma}_N^{-1} = \frac{a}{a+b^2}\vec{I} - \frac{b}{a^2+b^2}\vec{K} \ . \tag{21}$$

The duality theorem states that the effective resistivity tensor $\vec{\rho}_e^d$ for this N/I composite is identical to the effective *conductivity* tensor for the N/S composite. Thus, near p_c we have

$$\vec{\rho}_e^d = Aa(\Delta p)^{-s}\vec{I} + b\vec{K} \ . \tag{22}$$

This result states that the transverse magnetoresistance at *any field strength* diverges near p_c as $(\Delta p)^{-s}$, and also that the Hall voltage (and Hall coefficient) is *that of the normal metal* (again at any field strength) independent of p. The result for the Hall coefficient is identical to that of Juretschke, Landauer, and Swanson,[4] but is here generalized to *arbitrary* field strength.

A more remarkable result is obtained by considering the quantity conventionally considered as the magnetoresistance, namely, the fractional transverse increase in resistivity with magnetic field. From Eq. (22) this is just

$$\frac{[\rho_e^d(p,H)]_{xx} - \rho_e^d(p,0)}{\rho_e^d(p,0)} = \frac{a(H)-a(0)}{a(0)} \ , \tag{23}$$

which is *independent of p*. Thus, the magnetoresistance of an N/I composite anywhere above percolation is that of the *metallic* component at any field strength and does *not* depend on dilution. This surprising result is very much testable and should have interesting experimental consequences; note, however, that once again quantum effects, e.g., those which give rise to quantized Hall resistivities, will possibly alter these conclusions.

IV. DISCUSSION

The results proved here may be summarized as follows. (1) For a two-dimensional N/S composite with transverse magnetic field,

(a) $\rho(p,H) \propto (\Delta p)^{+s}$,

(b) $R_H(p,H) \propto (\Delta p)^{2s}$,

(c) $\rho(p,H=\infty)/\rho_e(p,0) \propto (\Delta p)^{-2s}$

near the percolation threshold. For a two-dimensional N/I composite above the percolation threshold, on the other hand,

(d) $\rho(p,H) \propto (\Delta p)^{-s}$,

(e) $R_H(p,H)$

is independent of p,

(f) $\rho(p,H)/\rho(p,0)$

is independent of p. Note the well-known result,[6] that the exponent s characterizing the divergence of the conductivity in a $2d$ N/S composite, equals the exponent t for the divergence of the resistivity in a $2d$ N/I composite, simply follows as a special case of the above, as does the low-field result for the Hall coefficient obtained by Juretschke, Landauer, and Swanson.[4]

It would be of great interest if the various predictions enumerated above could be checked experimentally. A suitable test could probably be carried out by construction of a specially prepared N/S or N/I film consisting of alternate cells of two types of material randomly distributed on the sites of a periodic lattice. The degree to which the theorems proved here are *not* satisfied will probably be a measure of the importance of the various fascinating macroscopic quantum phenomena in determining their transport properties.

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