

Conductivity due to classical phase fluctuations in a model for high- T_c superconductors

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We consider the real part of the conductivity, $\sigma_{1;\alpha\alpha}(\omega)$, arising from classical phase fluctuations in high- T_c superconductors. We show that $\int_0^\infty \sigma_{1;\alpha\alpha} d\omega \neq 0$ below the superconducting transition temperature T_c , provided there is some quenched disorder in the system. Furthermore, for a fixed amount of quenched disorder, this integral at low temperatures is proportional to the zero-temperature superfluid density, in agreement with experiment. We calculate $\sigma_{1;\alpha\alpha}(\omega)$ explicitly for a model of overdamped phase fluctuations.

Because of their high transition temperatures, small coherence lengths, and low superfluid densities, the cuprate superconductors are strikingly influenced by phase fluctuations of the superconducting order parameter. Such fluctuations are largely responsible for flux lattice melting¹ and vortex glass² transitions in a finite magnetic field. In addition, they strongly affect the zero-field transition,³ and possibly also the superconducting transition temperature itself in underdoped materials.⁴ Phase fluctuations also influence the transport properties of the high- T_c materials. For example, the finite-frequency conductivity shows a fluctuation-induced peak near T_c .⁵

Recent measurements in the most anisotropic high- T_c materials, such as $\text{Bi}_2\text{Sr}_2\text{Ca}_2\text{Cu}_2\text{O}_{8+\delta}$ (BSCCO), have found that the real part of the conductivity, $\sigma_1(\omega)$, not only has a peak near T_c but also remains quite large even far below T_c .⁶ Published measurements are available typically at frequencies in the 30–200 GHz range.^{6,7} In some cases, $\sigma_1(\omega, T)$ at such low temperatures exceeds the peak values observed near T_c .⁸ These large values occur over a broad concentration range varying from overdoped to optimally doped to underdoped. Finally, large low-temperature values of $\sigma_1(\omega, T)$ are correlated with a relatively large in-plane superfluid density $n_s(0)$ at $T=0$.

In this paper, we show that, in the presence of *quenched disorder*, classical phase fluctuations will produce a low-temperature background in $\sigma_1(\omega)$ which has many similarities to that observed in experiment.⁸ Specifically, we argue that in a sample with quenched disorder there must inevitably be a low-temperature background $\sigma_1(\omega, T)$ arising from classical phase fluctuations, whose frequency integral is related to the zero-temperature superfluid density, $n_s(0)$. While the exact frequency dependence of this background depends on the particular dynamics obeyed by the phase fluctuations, the frequency integral is independent of the dynamics.⁹ These fluctuations thus provide an alternative possible source of conductivity background, in addition to the gapless quasiparticles which should exist in a d -wave superconductor.¹⁰

In order to model the phase fluctuations, we adopt a *classical XY model* on a d -dimensional cubic lattice ($d=2$ or 3). We consider a lattice model of a superconductor, such that

each lattice point i is characterized by a phase θ_i . In the presence of a vector potential, the interaction Hamiltonian for this system is given by

$$H = - \sum_{\langle ij \rangle} J_{ij} \cos(\theta_i - \theta_j + A_{ij}), \quad (1)$$

where the sum runs over nearest-neighbor pairs; J_{ij} is the couplings between lattice points i and j . The gauge factor is $A_{ij} = (2\pi/\Phi_0) \int_i^j \mathbf{A} \cdot d\mathbf{l}$, where \mathbf{A} is the vector potential, $\Phi_0 = hc/q$ is the flux quantum, and q is the magnitude of the charge of a Cooper pair. The model can readily be generalized to treat anisotropic couplings;¹¹ the argument given below remains valid in that case also.

The diagonal components $n_{s;\alpha\alpha}$ of the superfluid density tensor are given by $n_{s;\alpha\alpha} = [(m^*c^2/(Vq^2))(\partial^2 F/\partial A_\alpha^2)]_{A_\alpha=0}$. Here m^* is the mass of a Cooper pair, F the Helmholtz free energy, V the sample volume, c the speed of light, and A_α is a fictitious uniform vector potential applied in the α direction in the presence of periodic boundary conditions in all three directions. Using $F = -k_B T \ln \Pi_{i=1}^N \int_0^{2\pi} d\theta_i \exp(-H/k_B T)$, where N is the number of lattice points, we obtain

$$n_{s;\alpha\alpha} = \frac{m^*c^2}{Vq^2} (\gamma_{1;\alpha} - \gamma_{2;\alpha}). \quad (2)$$

Here $\gamma_{1;\alpha} = [(2\pi/\Phi_0)a]^2 E_\alpha$, where a is the lattice constant, $\Phi_0 = hc/q$ is the flux quantum, and $E_\alpha = \langle \sum_{\langle ij \rangle \| \alpha} J_{ij} \cos \theta_{ij} \rangle$, the sum running over bonds in the direction α . Similarly, $\gamma_{2;\alpha} = [V^2/(c^2 k_B T)] \langle \mathcal{J}_\alpha(t)^2 \rangle$, where $\mathcal{J}_\alpha(t) = (a/V) \sum_{\langle ij \rangle \| \alpha} (qJ_{ij}/\hbar) \sin \theta_{ij}$, $\theta_{ij} = \theta_i - \theta_j$, the sums are carried out over distinct bonds in the α direction, and the triangular brackets denote a canonical thermal average.

The key point is to note that $\mathcal{J}_\alpha(t)$ is the volume-averaged supercurrent density in direction α . From this connection, we can use the Kubo formula,¹² in the classical limit, together with Eq. (2), to obtain an expression for $\sigma_{1;\alpha\alpha}(\omega)$, the real part of the long-wavelength conductivity in the direction α :

$$\begin{aligned}\sigma_{1;\alpha\alpha}(\omega) &= \frac{V}{k_B T} \int_0^\infty dt \cos(\omega t) \langle \mathcal{J}_\alpha(t) \mathcal{J}_\alpha(0) \rangle \\ &= \frac{a^2}{V k_B T} \int_0^\infty dt \cos(\omega t) \langle \mathcal{I}_\alpha(0) \mathcal{I}_\alpha(t) \rangle,\end{aligned}\quad (3)$$

where $I_{c;ij} = qJ_{ij}/\hbar$ and

$$\mathcal{I}_\alpha(t) = \sum_{(ij)|\alpha} I_{c;ij} \sin \theta_{ij}(t).\quad (4)$$

In order to relate $\sigma_{1;\alpha\alpha}(\omega)$ to the frequency-independent superfluid density $n_{s;\alpha\alpha}$, we need to integrate Eq. (3) over frequency, which leads to $\int_0^\infty \sigma_{1;\alpha\alpha}(\omega) d\omega = [V\pi/(2k_B T)] \langle \mathcal{J}_\alpha^2 \rangle$. Substituting this expression into Eq. (2), and using $\sigma_{\alpha\alpha}(-\omega) = \sigma_{\alpha\alpha}(\omega)$, we finally obtain

$$\begin{aligned}S_{1;\alpha} &\equiv \int_0^\infty \sigma_{1;\alpha\alpha}(\omega) d\omega \\ &= \frac{\pi c^2}{2} \left[\left(\frac{2\pi a}{\Phi_0} \right)^2 \frac{E_\alpha}{V} - \frac{n_{s;\alpha\alpha} q^2}{m^* c^2} \right].\end{aligned}\quad (5)$$

We first show that, for an *ordered* system, $S_{1;\alpha}^o(T=0) = 0$ at $T=0$. Suppose that $J_{ij} = J_\alpha$ for all bonds in the α direction. Then the phases θ_i are all equal and, with periodic boundary conditions, $E_\alpha = NJ_\alpha$. Similarly, in the limit $T \rightarrow 0$, $(\partial^2 F / \partial A_\alpha^2)_{A_\alpha=0} = (\partial^2 H / \partial A_\alpha^2)_{A_\alpha=0} = (2\pi a / \Phi_0)^2 NJ_\alpha$, from direct evaluation of the second derivative, with periodic boundary conditions. Hence, both terms on the right-hand side of Eq. (5) are equal, and $S_{1;\alpha}^o(T=0) = 0$.

We can use this result to obtain some analytical results for $S_{1;\alpha}(T=0)$ in the presence of quenched disorder. Since the right-hand side of Eq. (5) vanishes for the ordered case, $n_{s;\alpha\alpha}^o q^2 / m^* c^2 = (2\pi a / \Phi_0)^2 E_\alpha^o(0) / V$, where the superscript refers to the ordered system. Using this equivalence, we may rewrite $S_{1;\alpha}(T)$ as

$$S_{1;\alpha}(T) = \frac{\pi q^2 n_{s;\alpha\alpha}^o(0)}{2m^*} \left(\frac{E_\alpha(T)}{E_\alpha^o(0)} - \frac{n_{s;\alpha\alpha}(T)}{n_{s;\alpha\alpha}^o(0)} \right).\quad (6)$$

If the disordered system is unfrustrated (all $J_{ij} > 0$), all θ_i are still equal at $T=0$, and $E_\alpha(T=0)$ is controlled by the average bond strength \bar{J}_α : $E_\alpha(0) = N\bar{J}_\alpha$. In view of this fact, we choose the reference system to have $J_\alpha^o = \bar{J}_\alpha$, so that $E_\alpha(0)/E_\alpha^o(0) = 1$.

To further analyze this regime, we use a generalization of a theorem proved by Kirkpatrick,¹³ which maps the spin-wave stiffness constant of a random Heisenberg ferromagnet onto the conductance of a disordered conductance network, and generalized to the superfluid density of XY systems (the analog of the spin-wave stiffness constant) in Ref. 14. The generalized theorem, as applied to the present anisotropic geometry, states that

$$\frac{n_{s;\alpha\alpha}(T=0)}{n_{s;\alpha\alpha}^o(T=0)} = \frac{g_{\alpha\alpha}}{g_{\alpha\alpha}^o}.\quad (7)$$

Here $g_{\alpha\alpha}$ and $g_{\alpha\alpha}^o$ are components of the conductivity tensor of fictitious random conductance networks in which the bond conductances are J_{ij} and J_{ij}^o , the bond strengths of the real and reference systems.

Using this theorem, we can evaluate $S_{1;\alpha}(0)$ in an isotropic system for weak disorder. We retain our choice of the \bar{J}_α for the reference system. Thus $g_{\alpha\alpha}^o = \bar{g}$, the effective conductivity of a network made up of average conductances. For an isotropic conductance network in d dimensions,¹⁵ $g/\bar{g} = 1 - \langle (\delta g)^2 \rangle_{dis} / [\bar{g}^2 d] + \mathcal{O}((\delta g)^3)$, where $\delta g = g - \bar{g}$ and $\langle \dots \rangle_{dis}$ denotes an average over configurations of the quenched disorder. Hence, making use of Eq. (7), we obtain $n_{s;\alpha\alpha}(0) = n_{s;\alpha\alpha}^o(0) [1 - \langle (J - \bar{J})^2 \rangle_{dis} / \bar{J}^2 d]$, where $d=2$ or 3 is the dimensionality and \bar{J} is the average bond strength of a bond in this network.

Substituting this expression back into Eq. (6) gives our final result for the integrated fluctuation conductivity in the case of weak disorder:

$$\int_0^\infty \sigma_{1;\alpha\alpha}(\omega, T=0) d\omega = \frac{\pi q^2 n_{s;\alpha\alpha}(0) \langle (J - \bar{J})^2 \rangle_{dis}}{2m^* \bar{J}^2 d}.\quad (8)$$

Here we have used the fact that $n_{s;\alpha\alpha}(0)$ and $n_{s;\alpha\alpha}^o(0)$ are equal to lowest order in $\langle (J - \bar{J})^2 \rangle_{dis}$. The dimensions $d=2$ or $d=3$ would be appropriate for the limiting cases of no interplanar coupling ($d=2$) or a fully isotropic system ($d=3$); presumably a material such as $\text{Bi}_2\text{Sr}_2\text{Ca}_2\text{Cu}_2\text{O}_{8+\delta}$ would behave similarly to the $d=2$ case.

The result (8) has several similarities to experiments.^{6,8} Most strikingly, at fixed magnitude of the disorder parameter $\langle (J - \bar{J})^2 \rangle_{dis} / \bar{J}^2$, $S_{1;\alpha}(T=0)$ is predicted to scale with the low-temperature superfluid density, in agreement with experiment. Of course, the experiment is carried out at a specific frequency, while the relationship (8) applies to a frequency *integral*, but presumably $\sigma_{1;\alpha\alpha}(\omega)$ behave similarly for any given frequency. Note also that the result does not require the presence or absence of short-range order in the bond strength,¹⁵ but only that the bond distribution of bonds be macroscopically isotropic in d dimensions.

It is of interest to make a numerical estimate of the integral (8) for parameters appropriate for $\text{Bi}_2\text{Sr}_2\text{Ca}_2\text{Cu}_2\text{O}_{8+\delta}$. From the London equations, we may write $\pi q^2 n_{s;\alpha\alpha} / (2m^*) = c^2 / (8\lambda^2)$. If we take the in-plane penetration depth as 2000 Å and we use a low-temperature conductivity $\sigma_1(\omega, 0)$ of about $10^6 \Omega^{-1} \text{m}^{-1}$ at $\omega \sim 140$ GHz, as suggested in Ref. 8, and we further assume that this value remains roughly constant to low frequencies and cuts off at only slightly higher frequencies, then relation (8) could be satisfied for $\langle (J - \bar{J})^2 \rangle_{dis} / \bar{J}^2 \sim 10^{-1}$. This is at best an order-of-magnitude estimate, but it suggests at least that the relation (8) is not ruled out by experiment, since mean-square fluctuations in this range seem reasonable.

The present results are entirely independent of the specific dynamics of the XY model (1): *any* classical dynamics will give the same value for $S_{1;\alpha}(T)$. Nevertheless, in order to illustrate a possible behavior for $\sigma_1(\omega)$, we have approximated the dynamical response of the phase by the equations describing a coupled array of overdamped Josephson junc-

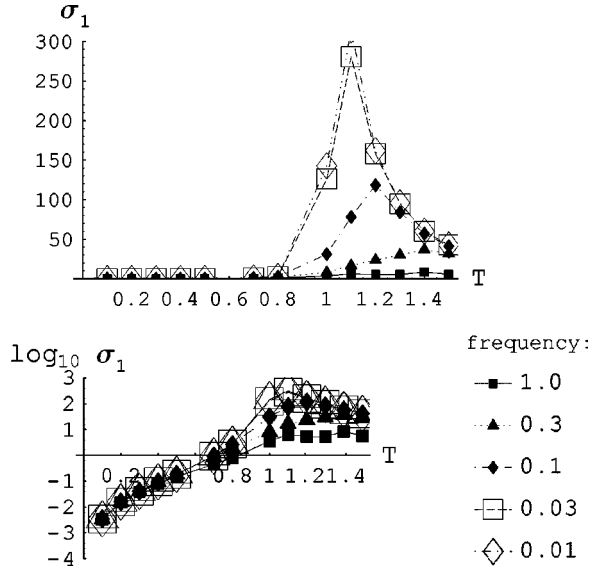


FIG. 1. Fluctuation conductivity $\sigma_1(\omega, T)$, plotted as a function of temperature T for several values of the frequency ω , calculated for an *ordered* 32×32 lattice with periodic boundary conditions. The inset shows the same results in a semi-log scale, demonstrating that $\sigma_1(\omega, T \rightarrow 0) \rightarrow 0$. Units are as described in the text, and lines are guides to the eye.

tions (see, for example, Ref. 16). In this picture, each lattice point is connected by an overdamped Josephson junction which carries three current contributions in parallel: a supercurrent $I_{c;ij} \sin \theta_{ij}$, a normal current through a shunt resistance R_{ij} , and a Langevin noise current $I_{L;ij}(t)$ representing the effects of thermal fluctuations.

We will consider the two-dimensional ($d=2$) case, for which the bond strengths J_{ij} vanish for $ij \parallel c$. In this limit, Eq. (3) reduces to [for the in-plane conductivity $\sigma_{1;xx}(\omega, T)$]

$$\sigma_{1;xx}(\omega) = \frac{1}{N_s a k_B T} \int_0^\infty dt \cos(\omega t) \langle \mathcal{I}_\alpha(0) \mathcal{I}_\alpha(t) \rangle, \quad (9)$$

where N_s is the number of lattice points in the layer, and the sum defining \mathcal{I}_α [Eq. (4)] runs over lattice points in a *single* layer. We have evaluated this average by solving the coupled Josephson equations, using the method described in Ref. 16.

Figures 1 and 2 show $\sigma_{1;xx}(\omega, T)$ for this model in two cases. In Fig. 1, all the critical currents and shunt resistances have the same values, I_c and R . In Fig. 2, the $I_{c;ij}$ are uniformly and independently distributed on the interval $(0, 2I_c)$, but all the R 's remain identical. In both cases, time is expressed in units of $\hbar/(qRI_c)$, frequency in units of qRI_c/\hbar , current in units of I_c , $k_B T$ in units of $\hbar I_c/q$, and therefore $\sigma_{1;xx}(\omega, T)$ in units of $1/(Ra)$.

The results for the two cases are strikingly different. For the *ordered* lattice, there is a fluctuation peak in $\sigma_1(\omega, T)$, more prominent at smaller frequencies, centered near the lattice phase-ordering temperature $T_c \sim 0.95 \hbar I_c/q$.¹⁷ (The peak is probably shifted away from this value by finite-size effects in our calculations.) For $T < T_c$, at all values of ω , $\sigma_1(\omega, T)$ falls off sharply towards a very small value at $T=0$, consistent with the prediction that $S_{1;\alpha\alpha}(0)=0$. [The decreasing character of $\sigma_1(\omega, T)$ is most evident in the semi-log plot.]

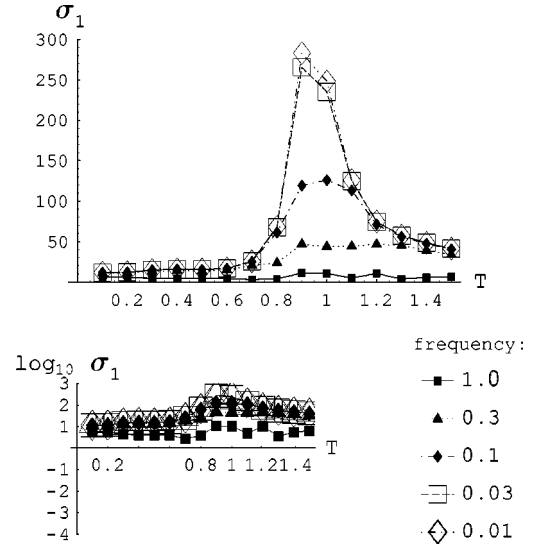


FIG. 2. $\sigma_1(\omega, T)$ plotted as a function of T for several frequencies in a *disordered* lattice (32×32 array), in which the critical current is uniformly and randomly distributed on the interval $(0, 2I_c)$. The inset shows same results in a semi-log scale. Lines are guides for eyes.

In the *disordered* lattice, $\sigma_1(\omega, T)$ still has a strong fluctuation peak, but in addition has a broad background for all $T < T_c$, which rolls off at larger frequencies. The frequency dependence is probably Lorentzian, as would be expected if the current-current correlation function in Eq. (3) decays exponentially in time. We expect that the relaxation time $\tau(T)$ should be of order $\hbar/(2eRI_c)$. The semi-log plot shows clearly that $\sigma_1(\omega, T)$ remains finite even as $T \rightarrow 0$.

The disorder-induced broad background in our calculations may appear rather small (much weaker than the peak near T_c). But this background is calculated in units of the strongly temperature-dependent coupling energy J . According to one model, for example,¹⁴ $J \propto \Delta^2$, where Δ is the mean-field energy gap, a quantity which decreases to zero at the mean-field transition temperature T_{c0} . When the temperature dependence of J is properly included, the fluctuation peak may well prove much smaller than the low-temperature background, especially when the low-temperature n_s is relatively large; this behavior would be in agreement with experiment.⁸

The disorder-induced background conductivity can be explained in a very simple way. In a disordered system, the effective superfluid density is reduced below its spatial average value by the disorder. Thus, the oscillator strength in the conductivity which is lost from the perfect-conductivity δ function at zero frequency must reappear at finite frequency. The result (8) is a special case of this general result.

There are several issues which we have not considered in the present work. For example, classical phase fluctuations might be expected to be frozen out by quantum effects at low T , as may happen in low- T_c superconductors.¹⁸ But in some high- T_c materials, the existence of nodal quasiparticles may provide a normal background which would impede this quantum freeze-out. In addition, zero-point phase oscillations may still give a contribution to the conductivity in the

quantum regime. It remains an open question to what temperature phase fluctuations persist in such materials as $\text{Bi}_2\text{Sr}_2\text{Ca}_2\text{Cu}_2\text{O}_{8+\delta}$.

In conclusion, we have demonstrated, both analytically and numerically, that in a superconductor with *quenched disorder*, the finite-frequency electrical conductivity $\sigma_1(\omega, T)$ remains nonzero at low temperatures if the order parameter has phase fluctuations which can be treated classically. If this assumption is satisfied, this result is quite general, and inde-

pendent of the dynamical equations obeyed by these fluctuations. For weak disorder, the frequency integral of this fluctuation conductivity scales proportional to the low-temperature superfluid density, in agreement with recent microwave experiments in $\text{Bi}_2\text{Sr}_2\text{Ca}_2\text{Cu}_2\text{O}_{8+\delta}$.

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