Block spins and Kadanoff derivation of scaling laws

Start from

$$\beta H = -\beta J \sum \frac{1}{N} S_i S_j - \beta H \sum \frac{1}{N} S_i$$

Ising model

$$= -K \sum \frac{1}{N} S_i S_j - h \sum \frac{1}{N} S_i$$

where $K = \beta J$; $h = \beta H$

Let $g_s(t, h) = \text{singular part of free energy per spin}$

Correlation length is $\xi(t)$
Lattice constant = $a$.
If we are near critical point, $a(t)$ is a response
Thus, if $a \ll b \ll \xi$, spins are then spins separated by $b$ are well correlated

Coarse grain by making a "block spin transformation."
Replace spins on a block of side \( l \) by a single block (which contains \( l^d \) spins).

Number of blocks = \( N l^{-d} \)

\( S_i = \text{block spin of block } i \)

\[
S_i = \frac{1}{m^d} \sum_{i \in \text{block } i} S_i
\]

Where \( m = \frac{1}{l^d} \sum_{i \in \text{block } i} \langle S_i \rangle = \text{magnetization of block } i \)

Note that \( \forall i \), \( \langle S_i \rangle = \pm 1 \)

Assumption I: Block spins interact only with nearest neighbor block spins and an effective external field.

Effective interaction is \( K_2 \epsilon h_2 \) with \( K_2 = \frac{K_1 h_2 l^d}{l^d} \).

Thus \( -\beta H_z = K_2 \sum_{i \neq i'} \frac{1}{2} S_i S_{i'} + h_z \sum_{i} S_i \)

lattice spacing is \( l_a \).

Note also that \( \xi = \frac{\xi}{l_a} = \text{correlation length} \) measured in units of new lattice spacing \( l_a \).

This implies that new system is farther from critical point than original one.

If \( \Delta = \frac{T - T_c}{T_c} \), then \( T_c \neq \Delta T_1 \),
Similarly
\[ h \sum_i S_i = h \sum_i \frac{S_i}{I} = h \frac{\sum_i S_i}{I} \]

So \( h \frac{S_i}{I} = h \frac{S_i}{I} \)

\( H_i \) has the same functional form as \( V \) original \( H \).

Thus \( N l^{-d} f_s(t, h) = N l^{-d} f_s(t, h) \)

since total singular free energy is the same.

Hence \( f_s(t, h) = l^d f_s(t, h) \). \( \text{[A]} \)

Now assume that \( t_0 = t l^y t \quad y_t > 0 \)

(Assumption III) \( h_0 = h l^y h \quad y_h > 0 \)

Will later find when this assumption is justified.

Plug back into eq. \( \text{[A]} \) to get

\[ f_s(t, h) = l^{-d} f_s(t l^y t, h l^y h) \]

Choose \( l = |t|^{-1/y_t} \) \( \text{[free to choose]} \)

\[ l^{y_t} |t| = 1 \]

\[ d/y_t \quad -y_t h/y_t \]

Then \( f_s(t, h) = |t|^{d/y_t} f(1, h l^y t) \)

Define \( \Delta = y_h/y_t \)
and \(2 - \alpha = d / y_t\).

Then \(f_s(t, h) = 1t^{2 - \alpha} F_f(h/t^\alpha)\)

where \(F_f(x) = f(1, x)\).

This is the start of the static scaling hypothesis

**Correlation functions:**

Now consider \(G(r, t) = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle\)

\(r \equiv \) distance between centers of blocks \(I\) and \(J\) in units of \(v_* t\).

Assume that take \(r_0 \gg 1\) — region where scaling makes sense.

How is \(G(r, t)\) related to \(G(r, t_0)\)?

\[
\langle \bar{S}_i \rangle = \int \frac{d^3 S_i}{6\pi^2 r_i^2} \left( e^{iS_i \cdot \mathbf{r}_i} + e^{-iS_i \cdot \mathbf{r}_i} \right)
\]

But \(S_i = \frac{1}{r_i} \int d^3 S_i\).
So \( G(r_e, t_e) \)
\[
= \frac{1}{|m_0|^2} \frac{1}{l^{2d}} \sum \sum \left[ \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \right]
\]

But \( h_0 = \hbar \left| m_0 \right|^d \) and

\[
\text{so } |m_0| = \frac{h_0}{\hbar} l^{-d} = l^{y_h - d}
\]

and thus
\[
G(r_e, t_e) = \frac{1}{l^{2(y_h - d)}} \frac{1}{l^{2d}} \sum \sum \left[ \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \right]
\]
\[
= \frac{1}{l^{2(y_h - d)}} \frac{1}{l^{2d}} l^d l^d \left[ \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \right]
\]
\[
= l^{2(d - y_h)} G(r, t)
\]

Including the dependence on \( h \), we get
\[
G\left( \frac{r}{l}, \frac{t}{l^{y_h}}, h l^{y_h} \right)
\]
\[
= l^{2(d - y_h)} G(r, t, h)
\]

Choose \( l = \frac{t}{y_h} \) to get
\[
G\left( \frac{r}{l^{1/y_h}}, \frac{t}{l^{y_h}}, h l^{y_h} \right) = G\left( \frac{r}{l^{1/y_h}}, \frac{t}{l^{y_h}}, h l^{y_h} \right)
\]

Define \( U_h \) \( G(n, t, h) \)
\[
G(r,t,h) = \frac{1}{4\pi} (r + \frac{1}{y t})^{2(d-y_h)} G(r,t,h)
\]

or

\[
G(r,t,h) = -2(d-y_h) (r + \frac{1}{y t})^{2(d-y_h)} G(r,t,1) \frac{1}{y t} \frac{y_h}{y t}
\]

where

\[
F_G(r + \frac{1}{y t}, h + \frac{y_h}{y t}) = (r + \frac{1}{y t})^{2(d-y_h)} G(r + \frac{1}{y t}, 1, h + \frac{y_h}{y t})
\]
Suppose that under $R_q$, the $K$'s become

$$[K'] = R_q [K] \quad l > 1$$

$R_q$ very complicated and nonlinear.

Two successive transformations with $l = l_1$, and

$$l = l_2$$

are equvil to one transform with $l = l_1 l_2$.

$$[K'] = R_{l_1} [K]$$

$$[K''] = R_{l_2} [K'] = R_{l_2} R_{l_1} [K]$$

Thus

$$R_q [K'] = R_{l_1} R_{l_2} [K]$$

"(semi group property)"

How to calculate $R_q$? No unique way.

Start with

$$Z_N[K] = \text{Tr} e^ {\beta G} \quad \beta G = -β H_N$$

$$g[K] = \frac{1}{N} \ln Z_N[K] = \text{free energy per spin}$$

RG transit reduces # of degrees of freedom

by $l^d$, leaving only block spin variables $S_i$.

$$g_N \{k, S_{i,j} \}$$

$$e^{g_N \{k, S_{i,j} \}} = \text{Tr} e^{\beta G} e^{g_N \{k, S_{i,j} \}}$$
\[ = \text{Tr}_{S_i}^2 P(S_i, S_i) \mathcal{N}(E_{K_i}, S_i) \]

\[ P(S_i, S_i) = \text{"projection operator"} \]

constructed so that

coarse-grained \( S_i \)'s

have same range of values as \( S_i \)'s.

E.g. suppose \( E_{K_i} \geq 0 \)

Use blocks with linear dimension \((2L+1)a\)

Ising model on square lattice

\[ P(S_i, S_i) = \prod_{i \in I} P(S_i - \text{sign} \left[ \sum_{i \in I} S_i \right]) \]

\( P(S_i, S_i) \) satisfies

(i): \( P(S_i, S_i) \geq 0 \)

(ii): \( P(S_i, S_i) \) somehow reflects symmetries of system

(iii): \( \sum_{S_i} P(S_i, S_i) = 1 \)

Similar to Kadanoff block transformation

but allows new \( K_i \)'s to be generated

during RG transformation
What happens during an RN6 transformation length scale increases by
\[ l, l^2, \ldots \rightarrow l^n \]

Coupling constants at \( n \)th stage are
\[ K_0^{(n)}, K_1^{(n)}, K_2^{(n)}, \ldots \]

⇒ trajectory in coupling constant space.

Set of all transformations
gives "RN6 flow."

Trajectory usually is attracted to fixed points

**Fixed points:** \( K^* = R^n[K^*] \)

Under \( R^n \),
\[ \xi[K] = \frac{\xi(K)}{l} \]

At fixed pt. \( \xi[K^*] = \xi[K^*]/l \)

⇒ \( \xi[K^*] = 0 \) or \( \infty \).

\( \xi[K^*] = \infty \) "critical fixed pt."
\( \xi[K^*] = 0 \) "trivial fixed point"

"Basin of attraction of \( K^* \): set of all \( K \)
which flow into \( K^* \) under \( R^n \)
Trivial fixed pts.: bulk phases

Critical fixed pts.: singular critical behavior

Local behavior of RG flow near a fixed point:

Suppose $K_n = K_n^* + \delta K_n$, $\delta K_n$ small.

Then $K_n' = K_n' [K] = K_n^* + \delta K_n$

And $K_n' (K_n^* + \delta K_1) (K_n^* + \delta K_2) \ldots$

$= K_n^* + \sum_{m} \frac{\partial K_n'}{\partial K_m} \delta K_m + \ldots$

So $\delta K_n' = \sum_{m} M_{nm} \delta K_m$

$M_{nm} = \left. \frac{\partial K_n'}{\partial K_m} \right|_{K = K^*}$

Linearized RG transformation
$M$ is real but not generally symmetric.

Left- and right-eigenvectors are not the same, nor are eigenvalues necessarily real.

However, they often are real, so in what follows we assume $M$ can be diagonalized and has real eigenvalues.

Now consider RG flow near fixed pt.

RG transformation $M(e^\lambda)$

(for Re $\lambda$)

Eigenvalues are called $\Lambda^{(\sigma)}$ and eigenvectors $\psi^{(n)}(\sigma)$.

$\sum_{m} M(e^\lambda)_{mn} \psi^{(m)}(\sigma) = \Lambda^{(\sigma)} \psi^{(n)}(\sigma)$ (Einstein summation convention)

$M(e^\lambda) M(e^{\lambda'}) = M(e^{\lambda + \lambda'})$ semigroup property

$\Rightarrow \Lambda^{(\sigma)} \Lambda^{(\sigma)} = \Lambda^{(\sigma)}$

E.g., if $\lambda = \lambda' = 1$ we get

$\Lambda^{(\sigma)} \Lambda^{(\sigma)} = \Lambda^{(\sigma)}$ or $\Lambda^{(\sigma)} \Lambda^{(\sigma)} = 1$
\[
\frac{d}{de^i} \left[ \Lambda^e_i (\sigma) \Lambda^i_e (\sigma) \right] = \frac{d}{de^i} \Lambda^e_i (\sigma)
\]

\[
= \Lambda^e_i (\sigma) \frac{d}{de^i} \Lambda^i_e (\sigma)
\]

Set \( e^i = 1 \):

\[
\Lambda^e_i (\sigma) \left( \frac{d}{de^i} \Lambda^i_e (\sigma) \right)_{e^i = 1} = \left[ \frac{d}{de^i} \Lambda^e_i (\sigma) \right]_{e^i = 1}
\]

\[
= e \left[ \frac{d}{de^i} \Lambda^e_i (\sigma) \right] = e \left[ \frac{d}{de^i} \Lambda^e_i (\sigma) \right]_{e^i = 1}
\]

\[
= e \left[ \frac{d}{de^i} \Lambda^e_i (\sigma) \right]_{e^i = e} \quad \text{since } e^i \text{ is just a dummy index.}
\]

Try \( \Lambda^e_i (\sigma) = e^i \gamma_0 \)

\[
\frac{d}{de^i} \Lambda^e_i (\sigma) = \gamma_0 \ e^i \gamma_0 - 1 \]

So last eq. becomes

\[
\sim e \gamma_0 \gamma_0 = e \gamma_0 e^i \gamma_0 - 1 = e \gamma_0 e^i \gamma_0 \checkmark
\]

So this solution works for any \( \gamma_0 \).

How does \( [\delta K] \) transform under \( M \)

\[
[\delta K] = \sum_{\sigma} \hat{\sigma} \ a^{(\sigma)} e^{(\sigma)} a^{(\sigma)} \quad \text{are coeffs.}
\]

vector of coupling constants
By assumed orthonormality, eigenvectors
\[ \phi^{(0)} = \phi^{(0)} : \delta K. \]

When we apply linearized RG transformation, we get
\[ \delta K' = M \delta K \]
\[ = M \sum_\sigma \phi^{(0)} \phi^{(0)} \]
\[ = M \sum_\sigma \phi^{(0)} \phi^{(0)} = \sum_\sigma \phi^{(0)} \phi^{(0)} \]

Thus \( \phi^{(0)} = \Lambda^{(0)} \phi^{(0)} \)

Some components of \( \delta K \) grow under \( M \)
while some shrink.

Let the eigenvalues be ordered by their absolute values, i.e.
\( |\Lambda_1| > |\Lambda_2| > |\Lambda_3| > \ldots \)

Then, there are three cases:

1. \( |\Lambda^{(0)}| > 1 \) i.e. \( \psi > 0 \) \( \phi^{(0)} \) grows as \( l \) increases.
2. \( |\Lambda^{(0)}| < 1 \) i.e. \( \psi < 0 \) \( \phi^{(0)} \) shrinks as \( l \) increases.
3. \( |\Lambda^{(0)}| = 1 \) i.e. \( \psi = 0 \) \( \phi^{(0)} \) unchanged with increasing \( l \).
Case (1): "relevant" eigenvalues/directions/eigenvectors
Case (2): "irrelevant"
Case (3): "marginal"

Relevant: flow away from fixed pt.
Irrelevant: into fixed pt.

Critical manifold: space whose dimension = 
# of relevant eigenvalues.

D. # of relevant eigenvalues = codimension = c of critical manifold
= dim. of coupling constant space - dimen. of critical manifold.

Global properties of RG flows:

Classification of fixed pts. by their codimension:

Codimension 0: no relevant trajectories.
   All trajectories flow into them.
   Hence called a sink.

Stable bulk phase:

E.g. 3D Ising model, NN FM model, coupling
   in an external field H has sinks.

RG at \( H = \pm \infty, T = 0 \)

Successive iterations bring system to one of these sinks.
Codimension 1:

1. Relevant directions

1st order phase transition

E.g. \( H=0, T<T_c \)

"discontinuity" fixed pt. - line of coexistence

"continuity" fixed pt. - e.g. \( H=0, T>T_c \)

bulk paramagnetic phase

Both types of fixed pts. unstable with respect to the sink:

Intentional external field causes RG flow to approach sink, not R\(\infty\) codimension fixed pts.

Codimension 2

Triple pt. \( \xi = 0 \)

Critical pt. \( \xi = 0 \infty \)

Here, the two relevant directions are the two variables which must be tuned to put system at the critical pt.

E.g. Ising model: need \( H=0 \) and \( T=T_c \).

Codimension > 2

\( \xi = 0 \) Multiple coexistence fixed pt.

\( \xi = 0 \infty \) Multicritical pt.
Example: RG for 2D triangular Ising model

\[ H = K \sum_{\langle ij \rangle} S_i S_j + h \sum_i S_i \]

\[ \delta C = -\beta H \]

\[ K = -\beta J \quad h = \beta H \]

Block spins

\[ s_i = \text{sign} (s_i^1 + s_i^2 + s_i^3) \]

called "majority rule"

\[ l = \sqrt{3} = \text{lattice const. renormalization} \]

Let \( \sigma_i = (s_i^1, s_i^2, s_i^3) \)

\[ s_i = +1 \text{ corresponds to } \uparrow \uparrow \uparrow \]

\[ s_i = -1 \text{ corresponds to } \downarrow \uparrow \downarrow \]

\[ s_i = \text{majority rule} \]
\[ e^{\mathcal{L}(\mathbf{S}_I)} = \mathbb{E} e^{\mathcal{L}_0(\mathbf{S}_I, \mathbf{\sigma})} \text{ such that the } \mathbf{\sigma}_i \text{'s allow} \]
\[ \text{correspond to the } \mathbf{S}_i \text{'s.} \]

Start with \( \mathcal{H} = 0 \).

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{V} \]

\[ \mathcal{H}_0 = K \sum_{i \neq j}^{\mathbf{S}_i \cdot \mathbf{S}_j} \]

\[ \mathcal{V} = K \sum_{i \in I} \sum_{j \in J} \sum_{i \in I} \sum_{j \in J} \mathbf{S}_i \cdot \mathbf{S}_j \]

We treat \( \mathcal{V} \) as a perturbation.

Let \( \langle A(\mathbf{S}_i) \rangle \) be the average of some \( A \) w.r.t. \( \mathcal{H}_0 \).

Then

\[ e^{\mathcal{L}(\mathbf{S}_I)} = \langle e^{\mathcal{V}} \rangle_0 \mathbb{E} e^{\mathcal{L}_0(\mathbf{S}_I, \mathbf{\sigma})} \]

\[ \sum_{\mathbf{\sigma}_I} e^{\mathcal{L}_0(\mathbf{S}_I, \mathbf{\sigma}_I)} = Z_0^M(\mathbf{K}) \quad M = \# \text{ of blocks} \]

\[ Z_0(\mathbf{K}) = \sum_{\mathbf{\sigma}_I} \exp \left\{ K \left[ \sum_{i \in I} \mathbf{S}_i \cdot \mathbf{S}_i + \sum_{i \neq j} \mathbf{S}_i \cdot \mathbf{S}_j \right] \right\} \]

These are the three spins in block \( I \) such that \( \mathbf{S}_I = \mathbf{\sigma}_I \).

\[ \mathbf{S}_I + \mathbf{\sigma}_I + \mathbf{\sigma}_I = \mathbf{S}_I \]
Last sum is \[ 3e^{-K} + e^{3K} \]

\[ S_1 = S_2 = 1 \quad S_1 = S_2 = S_3 = 1 \]

\[ S_3 = -1 \]

e tc.

Thus, \[ e^{\mathcal{U}(s_1)} = \langle e^v \rangle_0 Z_0^M (K) \]

How to compute \( \langle e^v \rangle_0 \)?

Well \[ \langle e^v \rangle_0 = 1 + \langle v^0 \rangle + \frac{1}{2} \langle v^2 \rangle + ... \]

Then \[ \ln \langle e^v \rangle_0 = \ln \left[ 1 + \langle e^{v^0} \rangle_0 - 1 \right] \]

\[ = \langle e^v \rangle_0 - 1 - \frac{1}{2} \left( \langle e^v \rangle_0 - 1 \right)^2 + ... \]

\[ = 1 + \langle v^0 \rangle + \frac{1}{2} \langle v^2 \rangle - 1 - \frac{1}{2} \left( \langle v^0 \rangle + \frac{1}{2} \langle v^2 \rangle \right) \]

\[ = \langle v^0 \rangle + \frac{1}{2} \left[ \langle v^2 \rangle - (\langle v^0 \rangle)^2 \right] \]

Thus \[ \langle e^v \rangle_0 \approx \exp \left\{ \ln \langle e^v \rangle_0 \right\} \]

\[ \approx \exp \left\{ \langle v^0 \rangle + \frac{1}{2} \left[ \langle v^2 \rangle - (\langle v^0 \rangle)^2 \right] + O(v^3) \right\} \]

Just keep \( \langle v^0 \rangle \). Then \( \mathcal{U}' = \langle v^0 \rangle + M \ln Z_0 (K) \)
\[ \langle V \rangle_0 = \sum_{I \neq J} \frac{V_{IJ}}{3} \]

\[ V_{IJ} = K (s^J_3)(s^I_{-1} + s^I_{2}) \]

So \[ V_{30} = K \langle s^J_3 \rangle \langle s^I_{-1} + s^I_{2} \rangle \]

\[ = 2K \langle s^J_3 \rangle \langle s^I_{-1} \rangle \]

But \[ \langle s^J_3 \rangle = \frac{1}{2z_0} s^J_3 e \]

\[ z_0 = \frac{3K}{2} + 3e^{-K} \]

For \( s^J = 1 \) \[ s^J_3 = \frac{1}{2z_0} \left( e^{-K} + e^K \right) = \frac{1}{2z_0} \left( 3K - K \right) \]

For \( s^J = -1 \) \[ s^J_3 = -\frac{1}{2z_0} \left( e^{-K} + e^K \right) = -\frac{1}{2z_0} \left( 3K - K \right) \]

Similarly \[ \langle s^I_{-1} \rangle = s^I_{-1} e^{-K} + e^{3K} \]

\[ \langle V \rangle_0 = 2K e^{2K} \sum_0 \langle s^I_{-1} \rangle s^I_{-1} s^J \]
where $\Phi(K) = \frac{e^{3K} - 1}{e^{3K} + 3e^{-K}}$

Thus

$\mathcal{E}'(S_I) = M \log Z_0(K) + K \frac{\partial S_I}{\partial K}$

$K' = 2K \Phi(K)$

Fixed point: $K^* = 2K^* \Phi(K^*)^2$

$\Rightarrow K^* = 0, \infty \text{ or } \frac{1}{\sqrt{2}} = \Phi(K^*)$

$\Rightarrow K^* = K_c = \frac{1}{4} \log (1 + 2\sqrt{2}) \approx 0.34$

$\Rightarrow K^* = K_c = \frac{1}{4} \ln 3 = 0.27 = \beta J$

Eigenvalue $\Lambda_+ = \left( \frac{\partial K'}{\partial K} \right)_{K_c} = 1.62$

Exact value should be $l$

$\sqrt{3} \sim 1.73 = l$
If you went to the next order and kept terms of form $\langle v^2 \rangle - \langle v_0^2 \rangle$, we wind up with interactions between second, third, ... nearest-neighbor blocks because

$$\langle v_0^2 \rangle \sim K^2 \langle \sum_{i,j} S_i S_j S_m S_n \rangle_0$$

$s_i, s_j$ in different blocks
and $s_m, s_n$ in different blocks.

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**Meaning of $\Lambda_t$:**

We define $\delta K' = \Lambda_t \delta K$

$$a' = \sqrt{3} a$$

So $\delta' = \frac{\delta}{\sqrt{3}}$

$$t' = \Lambda_t t$$

If $\delta \sim |t|^{-\psi}$

we have $|t'|^{-\psi} = \frac{1}{\sqrt{3}} |t|^{-\psi} = \Lambda_t^{-\psi} |t|^{-\psi}$

Thus $\Lambda_t^{-\psi} = \frac{1}{\sqrt{3}}$ or $\Lambda_t = (\sqrt{3})^\psi$

Onsager: $\sqrt{3} = 1 \Rightarrow \Lambda_t$ should be $\sqrt{3} = 1.73$
What if you have a magnetic field?

In this case, 

$$\hat{H} = K \sum_{i,j} \mathbf{S}_i \cdot \mathbf{S}_j + \hbar \sum_{i} \mathbf{S}_i$$

The renormalized magnetic field part is

$$\hat{H}' = \hbar \sum_{i} \mathbf{S}_i$$

and we have

$$e^{\hat{H}'}(\mathbf{S}_i \mathbf{S}_j) = \sum_{n} \mathbf{S}_n \mathbf{S}_n$$

where individual spins consistent with given block spins

$$\delta H = \hbar \sum_{i} \mathbf{S}_i \cdot \mathbf{S}_i$$

$$\sim \hbar \sum_{i} \mathbf{S}_i \cdot \mathbf{S}_i + \mathbf{S}_i \cdot \mathbf{S}_j$$

$$\langle \mathbf{S}_i \mathbf{S}_i \rangle \sim \mathbf{S}_i \cdot \mathbf{S}_i \Phi(K)$$

$$\Phi(K) = \frac{3K - K}{e^{3K} + 3e}$$

is internal perturbation of three-spin cluster.

$$\delta H = \hbar \Phi(K) \mathbf{S}_i = \Phi' \mathbf{S}_i$$

$$\Phi' = 3 \Phi(K)$$

$$\frac{d\Phi}{dH} = 3 \Phi(K)$$

$$\delta H = \frac{d\Phi}{dH}$$
\[ = \frac{3}{\sqrt{2}} \approx 2.12. \]

Exact result is \( \Omega_h \approx 2.8 \)