P631 Solutions to Midterm I

1. (a)

$$[\hat{x}, \hat{p}] = i\hbar$$

(b) The expectation value $\langle \hat{A} \rangle$ corresponds experimentally to carrying out one measurement each on a large number of identical systems and then averaging over all systems.

(c)(i) No, the system is not in a stationary state. It is in a superposition of energy eigenstates

(ii) No, $\hat{H}\Psi \neq (\text{constant})\cdot\Psi$

(iii) Yes, Ψ obeys $i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$

$$\text{LHS}: \; i\hbar\left[c_1\psi_{n_1}\left(\frac{-iE_{n_1}}{\hbar}\right)e^{-iE_{n_1}t/\hbar} + c_2\psi_{n_2}\left(\frac{-iE_{n_2}}{\hbar}\right)e^{-iE_{n_2}t/\hbar}\right]$$

$$= c_1E_{n_1}\psi_{n_1}e^{-iE_{n_1}t/\hbar} + c_2E_{n_2}\psi_{n_2}e^{-iE_{n_2}t/\hbar}$$

$$\text{RHS} = c_1\underbrace{\left(\hat{H}\psi_{n_1}\right)}_{E_{n_1}\psi_{n_1}}e^{-iE_{n_1}t/\hbar} + c_2\underbrace{\left(\hat{H}\psi_{n_2}\right)}_{E_{n_2}\psi_{n_2}}e^{-iE_{n_2}t/\hbar} \equiv \text{LHS}$$

(iv) Assuming $|c_1|^2 + |c_2|^2 = 1$ a measurement of energy will give E_{n_i} with probability $|c_{n_i}|^2$ for $i = 1$ or 2

2. $$\Psi(x,t=0) = \begin{cases} \sqrt{\frac{2}{a}} & \frac{a}{4} \le x \le \frac{3a}{4} \\ \phi & \text{otherwise} \end{cases}$$

(a) $$c_1 = \int \psi_1^* \, \Psi(x,0)\, dx = \int_{a/4}^{3a/4} \left(\sqrt{\frac{2}{a}} \sin\frac{\pi x}{a}\right) \sqrt{\frac{2}{a}}\, dx =$$

$$= \frac{2}{a}\left(-\cos\frac{\pi x}{a}\right)\cdot\frac{a}{\pi}\Bigg|_{a/4}^{3a/4} = \frac{2}{\pi}\Big[\underbrace{\cos\frac{\pi}{4} - \cos\frac{3\pi}{4}}_{2\cdot\frac{\sqrt{2}}{2}}\Big] = \frac{2\sqrt{2}}{\pi}$$

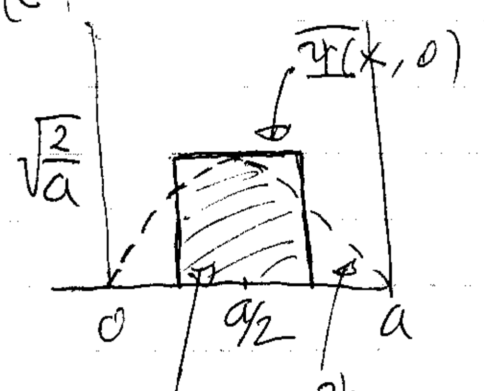
$$\boxed{P(E_1) = |c_1|^2 = \frac{8}{\pi^2}}$$

(b) $$c_2 = \int \psi_2^* \, \Psi(x,0)\, dx = \int_{a/4}^{3a/4} \left(\sqrt{\frac{2}{a}} \sin\frac{2\pi x}{a}\right) \sqrt{\frac{2}{a}}\, dx$$

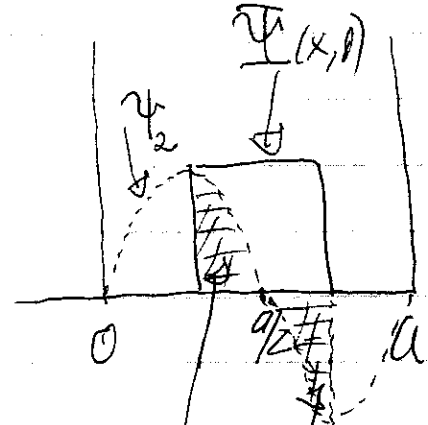
$$= \frac{2}{a}\left[-\cos\frac{2\pi x}{a}\cdot\frac{a}{2\pi}\right)\Bigg|_{a/4}^{3a/4} = \frac{1}{\pi}\left[\cos\frac{\pi}{2} - \cos\frac{3\pi}{2}\right] = 0$$

$$\boxed{P(E_2) = 0}$$

(c)



large overlap



>0 <0

positive and negative overlaps cancel exactly

3(a) $\int (\hat{a}\psi_a)^* \psi_b = \sqrt{\frac{m\omega}{2\hbar}} \int \left(\left[x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x} \right] \psi_a^* \right) \psi_b \, dx$

$= \sqrt{\frac{m\omega}{2\hbar}} \int \psi_a^* \left(\left[x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x} \right] \psi_b \right) dx$

where use was made of $\int \left(\frac{\partial}{\partial x}\psi_a^*\right)\psi_b = \cancel{\psi_a^*\psi_b \Big|_{-\infty}^{\infty}} \rightarrow 0$

$- \int \psi_a^* \left(\frac{\partial}{\partial x}\psi_b\right) dx$

So $\boxed{\int (\hat{a}\psi_a)^* \psi_b \, dx = \int \psi_a \left(\hat{a}^\dagger \psi_b\right) dx}$ (1)

(b)
Let $\psi_a \to \psi$ and $\psi_b \to \hat{a}\psi$ in above

$$\int (\hat{a}\psi)^* (\hat{a}\psi)\, dx = \int \psi^* \left(\hat{a}^\dagger \hat{a}\psi\right) dx \qquad (2)$$

QED

(c) If $\psi = \varphi_n$ then $\hat{a}^\dagger \hat{a}\varphi_n = \hat{N}\varphi_n = n\varphi_n$

Then $\int \varphi_{n-1}^* \varphi_{n-1}\, dx = \frac{1}{n} \int (\hat{a}\varphi_n)^* (\hat{a}\varphi_n)\, dx$

$= \frac{1}{n} \int \varphi_n^* \underbrace{\left(\hat{a}^\dagger \hat{a}\varphi_n\right)}_{n\varphi_n} dx = \int \varphi_n^* \varphi_n \, dx \equiv \underline{1}$