P631 Solutions to Set #8

1. (a) $\Phi(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{\pi}\right)^{1/4}\int_{-\infty}^{\infty} e^{-ax^2}\, e^{ik_0x}\, e^{-ikx}\,dx$

Use $\boxed{\int_{-\infty}^{\infty} e^{-Ay^2}\, e^{By}\,dy = \sqrt{\frac{\pi}{A}}\, e^{B^2/4A}}$ (1)

with $A \to a$, $B \to i(k_0-k)$ $\Rightarrow \frac{B^2}{4A} = \frac{-(k-k_0)^2}{4a}$

$$\Phi(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{\pi}\right)^{1/4}\sqrt{\frac{\pi}{a}}\, e^{-(k-k_0)^2/4a} = \boxed{\left(\frac{1}{2a\pi}\right)^{1/4} e^{-(k-k_0)^2/4a}}$$

$\Phi(k)$ = Gaussian centered around $k=k_0$.

(b) $\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int \Phi(k)\, e^{ikx}\, e^{-i\hbar k^2 t/2m}\,dk$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2a\pi}\right)^{1/4}\int e^{-\frac{1}{4a}(k^2-2kk_0+k_0^2)}\, e^{ikx}\, e^{-i\hbar k^2 t/2m}\,dk$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2a\pi}\right)^{1/4} e^{-k_0^2/4a}\int e^{-k^2\left[\frac{1}{4a}+i\hbar t/2m\right]}\, e^{k\left[\frac{k_0}{2a}+ix\right]}\,dk$$

Use (1) again with $\begin{cases} A = \frac{1}{4a} + \frac{i\hbar t}{2m} \equiv \frac{1}{4a}\left(1+\frac{it}{\tau}\right) \\ B = \frac{k_0}{2a} + ix = i\left(x - i\frac{k_0}{2a}\right)\end{cases}$

$\left(\tau \equiv \frac{m}{2a\hbar}\right)$

$$\Rightarrow B^2/4A = \frac{-\left(x-ik_0/2a\right)^2}{\frac{1}{a}\left(1+\frac{it}{\tau}\right)}$$

$$\boxed{\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\sqrt{1+it/\tau}}\, e^{-k_0^2/4a}\, e^{-a\left[\left(x-ik_0/2a\right)^2/\left(1+it/\tau\right)\right]}}$$ (2)

$|\Psi(x,t)|^2 = ?$

- Look separately at each factor in (2)

(i) $$\frac{1}{\sqrt{1+it/\tau}}\ \frac{1}{\sqrt{1-it/\tau}} = \boxed{\frac{1}{\sqrt{1+t^2/\tau^2}}} \quad (3)$$

(ii) $$e^{-a\left[(x-ik_0/2a)^2/(1+it/\tau)\right]}\ e^{-a\left[(x+ik_0/2a)^2/(1-it/\tau)\right]}$$

Add the 2 exponents:

$$-a\left\{\frac{(x-ik_0/2a)^2}{(1+it/\tau)} + \frac{(x+ik_0/2a)^2}{(1-it/\tau)}\right\} =$$

$$= \frac{-a}{(1+t^2/\tau^2)}\left\{(x-ik_0/2a)^2(1-it/\tau) + (x+ik_0/2a)^2(1+it/\tau)\right\}$$

$$= \frac{-2a}{(1+t^2/\tau^2)}\left\{x^2 - \frac{k_0^2}{4a^2} + \left(\frac{-ik_0x}{a}\right)\left(-i\frac{t}{\tau}\right)\right\}$$

$$= \frac{-2a}{(1+t^2/\tau^2)}\left\{x^2 - \frac{k_0^2}{4a^2} - \frac{xk_0t}{a\tau}\right\}$$

add and subtract $\dfrac{k_0^2t^2}{4a^2\tau^2}$

(ii) $$\to \frac{-2a}{(1+t^2/\tau^2)}\left\{\left(x-\frac{k_0t}{2a\tau}\right)^2 - \frac{k_0^2}{4a^2}(1+t^2/\tau^2)\right\} =$$

$$= \boxed{\frac{-2a}{(1+t^2/\tau^2)}\left(x-\frac{k_0t}{2a\tau}\right)^2 + \frac{k_0^2}{2a}} \quad (4)$$

(iii) $\left[\left(\frac{2a}{\pi}\right)^{1/4} e^{-k_0^2/4a}\right]^2 = \boxed{\sqrt{\frac{2a}{\pi}}\, e^{-k_0^2/2a}}$ (5)

Multiply (3) and (5) and exponential of (4).

$$\boxed{|\Psi(x,t)|^2 = \sqrt{\frac{2a}{\pi}}\,\frac{1}{\sqrt{1+t^2/\tau^2}}\, e^{-2a(x-k_0t/2a\tau)^2/(1+t^2/\tau^2)}} \quad (6)$$

Similar to stationary Gaussian with

$x \to x - \frac{k_0 t}{2a\tau} \equiv x - v_0 t$, $\boxed{v_0 = \frac{k_0}{2a\tau} = \frac{\hbar k_0}{m}}$

Center of Gaussian moves with v_0.

(c)

$\langle \hat{x} \rangle = \int x |\Psi|^2 dx$

Let $x - v_0 t \equiv y$ or $x = y + v_0 t$

$$\langle \hat{x} \rangle = \int (y + v_0 t)\sqrt{\frac{2a}{\pi}}\,\frac{1}{\sqrt{1+t^2/\tau^2}}\, e^{-2ay^2/(1+t^2/\tau^2)}\, dy$$

this term integrates to zero ($y \to -y$ symmetry)

One is left with a simple Gaussian integral

$$\int_{-\infty}^{\infty} e^{-qy^2} dy = \sqrt{\frac{\pi}{q}} \quad \text{for } q = \frac{2a}{(1+t^2/\tau^2)}$$

$\Rightarrow \boxed{\langle \hat{x} \rangle = v_0 t}$ $\qquad \boxed{\langle \hat{p} \rangle = m\frac{d}{dt}\langle \hat{x} \rangle = m v_0 = \hbar k_0}$