P631 Solutions to Set #7.

1. (Griffiths Prob. 2.41)

(a) From Griffiths eq. [2.85] one has $\left(A_0 \equiv \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\right)$

$$\varphi_0 = A_0 e^{-\xi^2/2}, \quad \varphi_1 = A_0 \sqrt{2}\,\xi e^{-\xi^2/2}, \quad \varphi_2 = A_0 \frac{1}{\sqrt{2}}(2\xi^2-1)e^{-\xi^2/2}$$

$$\Psi(x,t=0) = A(1-2\xi)^2 e^{-\xi^2/2} = A(1-4\xi+4\xi^2)e^{-\xi^2/2} \quad (1)$$

We want to rewrite as

$$\Psi(x,0) = c_0\varphi_0 + c_1\varphi_1 + c_2\varphi_2 = \\ = A_0\left[c_0 + c_1\sqrt{2}\,\xi + c_2\frac{1}{\sqrt{2}}(2\xi^2-1)\right]e^{-\xi^2/2} \quad (2)$$

Equate coefficients of powers of ξ in (1) and (2)

$$\begin{cases} \xi^0: A = A_0\left(c_0 - \frac{c_2}{\sqrt{2}}\right) \\ \xi^1: -4A = \sqrt{2}\,c_1 A_0 \Rightarrow \boxed{c_1 = -\frac{4}{\sqrt{2}}\frac{A}{A_0}} \\ \xi^2: 4A = A_0 c_2\sqrt{2} \Rightarrow \boxed{c_2 = \frac{4}{\sqrt{2}}\frac{A}{A_0}} \end{cases}$$

$$c_0 = \frac{A}{A_0} + \frac{c_2}{\sqrt{2}} = \frac{A}{A_0}[1+2] = \boxed{3\frac{A}{A_0}}$$

Normalize: $|c_0|^2 + |c_1|^2 + |c_2|^2 = 1$

$$\left(\frac{A}{A_0}\right)^2[9+8+8] = \left(\frac{A}{A_0}\right)^2 25 \equiv 1$$

$$\boxed{\frac{A}{A_0} = \frac{1}{5}} \quad \boxed{c_0 = \frac{3}{5}} \quad \boxed{c_1 = -\frac{2\sqrt{2}}{5}} \quad \boxed{c_2 = \frac{2\sqrt{2}}{5}}$$

$$\Psi(x,0) = \frac{1}{5}\left[3\varphi_0 - 2\sqrt{2}\,\varphi_1 + 2\sqrt{2}\,\varphi_2\right]$$

In general, if $\Psi = \sum_n c_n \varphi_n$ then

$$\langle H \rangle = \sum_n |c_n|^2 E_n$$

So we have $\langle \hat{H} \rangle = \frac{1}{25}\left[9E_0 + 8E_1 + 8E_2\right] =$

$$= \frac{1}{25}\hbar\omega\left[9\,\tfrac{1}{2} + 8\,\tfrac{3}{2} + 8\,\tfrac{5}{2}\right] = \boxed{\frac{73}{50}\hbar\omega}$$

(b) $\Psi(x,t) = \frac{1}{5}\left[3\varphi_0 e^{-i\omega t/2} - 2\sqrt{2}\,\varphi_1 e^{-i3\omega t/2} + 2\sqrt{2}\,\varphi_2 e^{-i5\omega t/2}\right]$

$$= \frac{1}{5} e^{-i\omega t/2}\left[3\varphi_0 - 2\sqrt{2}\,\varphi_1 e^{-i\omega t} + 2\sqrt{2}\,\varphi_2 e^{-i2\omega t}\right]$$

We know that $\frac{1}{5}\left[3\varphi_0 - 2\sqrt{2}\,\varphi_1 + 2\sqrt{2}\,\varphi_2\right]$

$$\propto (1-2\xi)^2 e^{-\xi^2/2} = (1 - 4\xi + 4\xi^2)\, e^{-\xi^2/2}$$

We want to switch sign of (-4ξ) term
i.e. of the φ_1 term.

$\Rightarrow$ want $e^{-i\omega t} = -1$, $e^{-i2\omega t} = \left(e^{-i\omega t}\right)^2 = 1$

Since $e^{-i\pi} = \cos\pi - i\sin\pi = -1$ want $\omega T = \pi$

$$\boxed{T = \frac{\pi}{\omega}}$$

2. (Griffiths Prob. 2.21)

(a) $1 = \int_{-\infty}^{\infty} |\Psi|^2 dx = 2|A|^2 \int_0^{\infty} e^{-2ax} dx = \frac{|A|^2}{a} e^{-2ax}\Big|_{\infty}^{0}$

$= \frac{1}{a}|A|^2$

Use has been made of

$$\int_{-\infty}^{0} e^{-2a|x|} dx = \int_{-\infty}^{0} e^{2ax} dx \underset{y \equiv -x}{=} \int_{\infty}^{0} e^{-2ay} (-dy) = \int_0^{\infty} e^{-2ay} dy$$

So, $\boxed{A = \sqrt{a}}$

(b) $\phi(k) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-a|x|} dx$

Write $e^{-ikx} = \cos kx - i \sin kx$ and take into account that

$\cos kx$ & $e^{-a|x|}$ are symmetric

while $\sin(kx)$ is an antisymmetric fcn. of x.

$\Rightarrow \phi(k) = 2\frac{A}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} \cos kx \, dx =$

$= 2\frac{A}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right) dx$

$= \frac{A}{\sqrt{2\pi}} \left[\frac{e^{(ik-a)x}}{ik-a} + \frac{e^{-(ik+a)x}}{-(ik+a)} \right]\Bigg|_0^{\infty} =$

$= \frac{A}{\sqrt{2\pi}} \left[\frac{1}{a-ik} + \frac{1}{a+ik} \right] = \boxed{\sqrt{\frac{a}{2\pi}} \frac{2a}{a^2+k^2}}$

(c) $\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \phi(k)\, e^{ikx}\, e^{-iE_k t/\hbar}\, dk$

$$\boxed{\Psi(x,t) = \frac{a^{3/2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{(a^2+k^2)}\, e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)}\, dk}$$

(d)

For large "a" $\phi(k) \sim \sqrt{\frac{2}{\pi a}}$ is broad and indep of k, whereas $\Psi(x,0)$ is sharply peaked near $x=0$.
So position well defined and momentum ill defined.

For small "a", $\Psi(x,0)$ is broad whereas $\phi(k) \sim \sqrt{\frac{2a^3}{\pi}}\, \frac{1}{k^2}$ is sharply peaked
Position ill defined but momentum well defined