1. $\boxed{\text{Eq. (5)}}$: $x\varphi_1 = x A_0 \sqrt{2}\, \xi\, e^{-\xi^2/2} = \boxed{\sqrt{\frac{\hbar}{m\omega}} A_0 \sqrt{2}\, \xi^2 e^{-\xi^2/2}}$

with $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}\, x$, $A_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

On the other hand $\varphi_0 = A_0 e^{-\xi^2/2}$, $\varphi_2 = \frac{1}{\sqrt{2}} A_0 (2\xi^2 - 1)\, e^{-\xi^2/2}$

So, $\boxed{\text{Eq. (6)}}$: $\sqrt{\frac{\hbar}{2m\omega}} (\varphi_0 + \sqrt{2}\, \varphi_2) =$

$$= \sqrt{\frac{\hbar}{2m\omega}} A_0 \{1 + (2\xi^2 - 1)\} e^{-\xi^2/2} = \boxed{\sqrt{\frac{2\hbar}{m\omega}} A_0 \xi^2 e^{-\xi^2/2}}$$

same as above.

---

2. $\Psi(x,t) = \frac{1}{\sqrt{2}} \left(\varphi_7 e^{-iE_7 t/\hbar} + \varphi_8 e^{-iE_8 t/\hbar}\right)$

$E_7/\hbar =$, $E_8/\hbar = \frac{17}{2}\omega$

$\hat{a}\Psi(x,t) = \frac{1}{\sqrt{2}} \left(\sqrt{7}\, \varphi_6\, e^{-iE_7 t/\hbar} + \sqrt{8}\, \varphi_7\, e^{-iE_8 t/\hbar}\right)$

$\int \Psi^* \hat{a} \Psi\, dx = \langle \hat{a} \rangle =$

$$= \frac{1}{2} \int \left[\varphi_7 e^{iE_7 t/\hbar} + \varphi_8 e^{iE_8 t/\hbar}\right] \left[\sqrt{7}\, \varphi_6 e^{-iE_7 t/\hbar} + \sqrt{8}\, \varphi_7 e^{-iE_8 t/\hbar}\right] dx$$

$$= \frac{1}{2} \sqrt{8}\, e^{-i(E_8 - E_7)t/\hbar} = \boxed{\sqrt{2}\, e^{-i\omega t}}$$

Since $\langle \hat{a} \rangle$ is complex in general, $\hat{a}$ cannot corresp. to a physical observable (whose expectation values must be real)

3. (Griffiths Prob. 2.13)

(a) $\Psi(x,0) = A[3\varphi_0 + 4\varphi_1]$

$$\int |\Psi|^2 dx = |A|^2 \int (3\varphi_0^* + 4\varphi_1^*)(3\varphi_0 + 4\varphi_1)\,dx =$$

$$= |A|^2 (9+16) = 25|A|^2 \equiv 1$$

$$\Rightarrow A = \frac{1}{5}, \quad \boxed{\Psi(x,0) = \frac{1}{5}[3\varphi_0 + 4\varphi_1]}$$

(b) $\Psi(x,t) = \frac{1}{5}\left[3\varphi_0 e^{-iE_0 t/\hbar} + 4\varphi_1 e^{-iE_1 t/\hbar}\right]$

$$= \boxed{\frac{1}{5}\left[3\varphi_0 e^{-i\omega t/2} + 4\varphi_1 e^{-i3\omega t/2}\right]}$$

where $E_0 = \frac{\hbar\omega}{2}$, $E_1 = \frac{3\hbar\omega}{2}$

$$|\Psi(x,t)|^2 = \frac{1}{25}\left(3\varphi_0 e^{i\omega t/2} + 4\varphi_1 e^{i3\omega t/2}\right) *$$
$$\left(3\varphi_0 e^{-i\omega t/2} + 4\varphi_1 e^{-i3\omega t/2}\right)$$

$$= \frac{1}{25}\left(9\varphi_0^2 + 16\varphi_1^2 + 12\varphi_0\varphi_1\left[e^{i\omega t} + e^{-i\omega t}\right]\right)$$

$$\boxed{|\Psi(x,t)|^2 = \frac{1}{25}\left(9\varphi_0^2 + 16\varphi_1^2 + 24\varphi_0\varphi_1 \cos\omega t\right)}$$

(c) $\langle \hat{x} \rangle = \int \Psi^*(x,t)\, \hat{x}\, \Psi(x,t)\, dx$

Use $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger)$, $\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}} (\hat{a}^\dagger - a)$

So $$\boxed{\hat{x}\Psi(x,t) = \sqrt{\frac{\hbar}{2m\omega}}\, \frac{1}{5} \left[4\varphi_0 e^{-i3\omega t/2} + 3\varphi_1 e^{-i\omega t/2} + 4\sqrt{2}\,\varphi_2 e^{-i3\omega t/2} \right]}$$

$$\langle \hat{x} \rangle = \sqrt{\frac{\hbar}{2m\omega}}\, \frac{1}{25} \int \left[3\varphi_0 e^{i\omega t/2} + 4\varphi_1 e^{i3\omega t/2} \right] * \left[4\varphi_0 e^{-i3\omega t/2} + 3\varphi_1 e^{-i\omega t/2} + 4\sqrt{2}\,\varphi_2 e^{-i3\omega t/2} \right] dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\, \frac{1}{25} \left(12 e^{-i\omega t} + 12 e^{i\omega t} \right) =$$

$$\boxed{\langle \hat{x} \rangle = \sqrt{\frac{\hbar}{2m\omega}}\, \frac{24}{25} \cos(\omega t)}$$

$$\langle \hat{p} \rangle = i\sqrt{\frac{\hbar m\omega}{2}}\, \frac{1}{25} \int \left[3\varphi_0 e^{i\omega t/2} + 4\varphi_1 e^{i3\omega t/2} \right] * \left[3\varphi_1 e^{-i\omega t/2} + 4\sqrt{2}\,\varphi_2 e^{-i3\omega t/2} - 4\varphi_0 e^{-i3\omega t/2} \right] dx$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\, \frac{1}{25} \left(12 e^{i\omega t} - 12 e^{-i\omega t} \right) = \boxed{-\sqrt{\frac{\hbar m\omega}{2}}\, \frac{24}{25} \sin(\omega t)}$$

If φ_1 were replaced by φ_2 one would have had $\cos(2\omega t)$ and $\sin(2\omega t)$.

$$\boxed{\left\langle -\frac{\partial \hat{V}}{\partial x} \right\rangle = -m\omega^2 \langle \hat{x} \rangle = -\omega\sqrt{\frac{\hbar m\omega}{2}}\, \frac{24}{25} \cos(\omega t)}$$

One sees that $\frac{d\langle \hat{p} \rangle}{dt} = \left\langle -\frac{\partial \hat{V}}{\partial x} \right\rangle$ is satisfied.

(d) One can get $E_0 = \frac{\hbar\omega}{2}$ with probab. $\frac{9}{25}$

and $E_1 = \frac{3\hbar\omega}{2}$ with probab. $\frac{16}{25}$