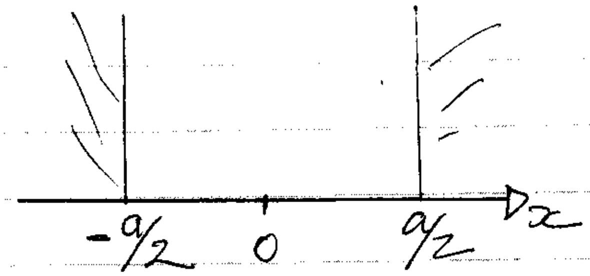
P631 Solutions to Set #5

1. Particle in symmetric box
$-\frac{a}{2} \leq x \leq a/2$



$$V(x) = \begin{cases} 0 & -\frac{a}{2} \leq x \leq \frac{a}{2} \\ \infty & \text{otherwise} \end{cases}$$

Since $V(-x) = V(x)$ one can solve separately for even and odd solutions.

$\boxed{\text{even}}$

$$\hat{H}\psi = E\psi \longrightarrow \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi$$

even solution $\boxed{\psi_{even} = B\cos kx}$

Impose boundary condition at $x = a/2$

$$\psi(a/2) = B\cos\frac{ka}{2} = 0 \Rightarrow k\frac{a}{2} = \frac{(2\ell-1)\pi}{2}$$

Since $k^2 = \frac{2mE}{\hbar^2} \Rightarrow$
$$\boxed{E = E_{2\ell-1} = \frac{\hbar^2\pi^2(2\ell-1)^2}{2ma^2}} \quad (1)$$

$$\boxed{\psi_{2\ell-1} = \sqrt{\frac{2}{a}}\cos\left[\frac{(2\ell-1)\pi}{a}x\right]} \quad (2)$$

$\ell = 1, 2, 3, \cdots$

$\boxed{\text{odd}}$ $\boxed{\psi_{odd} = A \sin kx}$

Again B.C. at $x = \frac{a}{2}$ $\Rightarrow$ $\sin k\frac{a}{2} = 0$ $\Rightarrow$ $k\frac{a}{2} = \ell\pi$

$\boxed{k = \frac{2\ell\pi}{a}}$

$$\boxed{E_{2\ell} = \frac{\hbar^2\pi^2(2\ell)^2}{2m a^2}} \qquad (3)$$

$$\boxed{\psi_{2\ell} = \sqrt{\frac{2}{a}}\, \sin\left(\frac{2\ell\pi}{a}x\right)} \qquad (4)$$

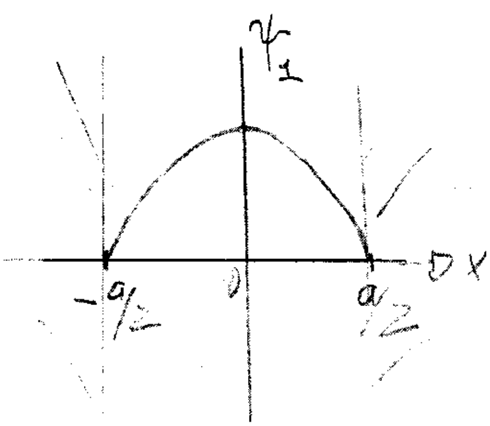
$\ell = 1, 2, \ldots$

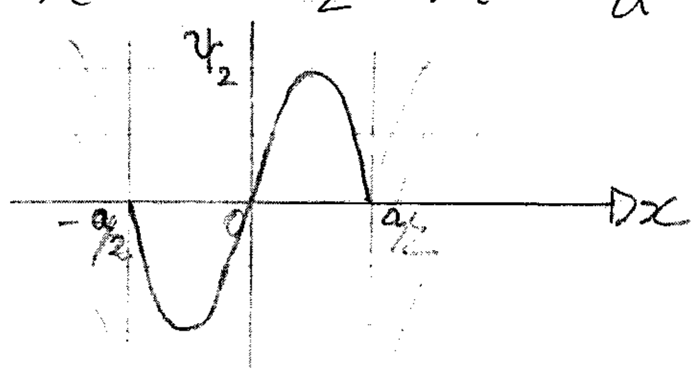
The $\left[E_{2\ell-1} \text{ of } (1) + E_{2\ell} \text{ of } (3)\right]$ for $\ell = 1, 2, \ldots$

reproduce the $E_n = \frac{\hbar^2\pi^2 n^2}{2ma^2}$, $n = 1, 2, \ldots$

of the $0 \le x \le a$ box

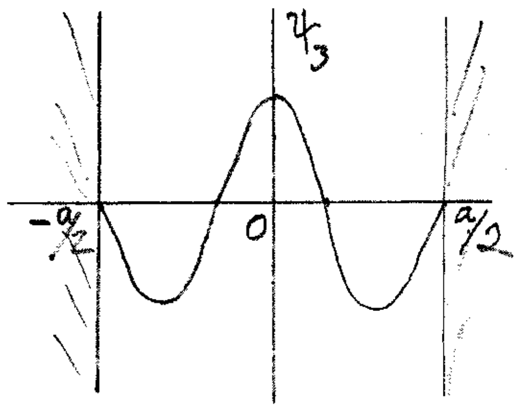
For $\ell = 1$ one has $\begin{cases} \psi_{2\ell-1} \rightarrow \psi_1 = \sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right) \\ \psi_{2\ell} \rightarrow \psi_2 = \sqrt{\frac{2}{a}} \sin\frac{2\pi}{a}x \end{cases}$

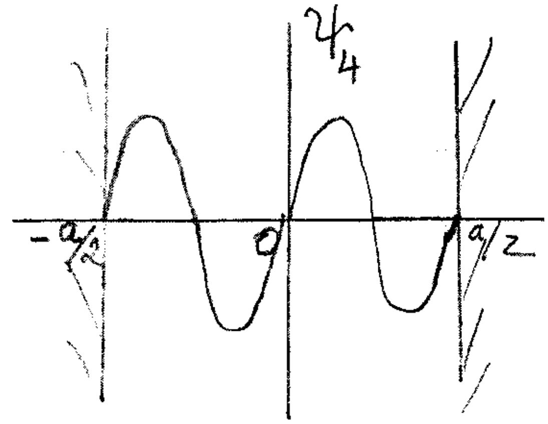




Similarly for $\ell = 2$

$$\psi_3 = \sqrt{\frac{2}{a}} \cos\left(\frac{3\pi}{a}x\right) \qquad \psi_4 = \sqrt{\frac{2}{a}} \sin\left(\frac{4\pi}{a}x\right)$$





These 4 eigenfcns are the same as the $\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$, $n = 1, 2, 3, 4$ with $0 \leq x \leq a$.

up to overall sign for ψ_2. This overall sign (overall phase in general) does not affect any physics results.

2. (a) (Griffiths Prob. 2.12)

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^\dagger)$$

$$\langle \hat{x} \rangle = \int \varphi_n^* \sqrt{\frac{\hbar}{2m\omega}} \,(\underbrace{\hat{a}\varphi_n}_{\propto \varphi_{n-1}} + \underbrace{\hat{a}^\dagger \varphi_n}_{\propto \varphi_{n+1}})\, dx = \boxed{0}$$

orthogonality of $\{\varphi_n\}$

$$\hat{x}^2 = \frac{\hbar}{2m\omega}(\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) = \frac{\hbar}{2m\omega}(\hat{a}^2 + \hat{a}^{\dagger 2} + \underbrace{\hat{a}\hat{a}^\dagger}_{=\hat{a}^\dagger\hat{a} + 1} + \hat{a}^\dagger \hat{a})$$

$$= \frac{\hbar}{2m\omega}(\hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}^\dagger \hat{a} + 1)$$

$$\langle \hat{x}^2 \rangle = \int \varphi_n^* \frac{\hbar}{2m\omega}\Big[\underbrace{\hat{a}^2\varphi_n}_{\propto \varphi_{n-2}} + \underbrace{\hat{a}^{\dagger 2}\varphi_n}_{\propto \varphi_{n+2}} + \underbrace{2\hat{a}^\dagger \hat{a}\varphi_n}_{=2n\varphi_n} + \varphi_n\Big]dx$$

$$\boxed{\langle \hat{x}^2 \rangle = \frac{\hbar}{2m\omega}(2n+1)}$$

$$\hat{p} = \sqrt{\frac{\hbar m\omega}{2}}\; i\,(\hat{a}^\dagger - \hat{a}), \qquad \langle \hat{p} \rangle = \boxed{0}$$

$$\langle \hat{p}^2 \rangle = \frac{-\hbar m\omega}{2}\int \varphi_n^* \Big[\underbrace{\hat{a}^{\dagger 2}\varphi_n}_{\propto \varphi_{n+2}} + \underbrace{\hat{a}^2\varphi_n}_{\propto \varphi_{n-2}} \underbrace{- 2\hat{a}^\dagger\hat{a}\varphi_n}_{-2n\varphi_n} - \varphi_n\Big]dx$$

$$\boxed{\langle \hat{p}^2 \rangle = \frac{\hbar m\omega}{2}(2n+1)} \qquad \langle \hat{T} \rangle = \left\langle \frac{\hat{p}^2}{2m} \right\rangle = \frac{\hbar\omega}{4}(2n+1)$$

$$\boxed{\langle \hat{T} \rangle = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right)}$$

same as $\langle \hat{V} \rangle$

$$\begin{cases} \sigma_x = \sqrt{\langle \hat{x}^2 \rangle - \langle x \rangle^2} = \sqrt{\dfrac{\hbar}{m\omega}} \left(n + \dfrac{1}{2}\right)^{1/2} \\ \sigma_p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2} = \sqrt{\hbar m\omega} \left(n + \dfrac{1}{2}\right)^{1/2} \end{cases}$$

So,

$$\boxed{\sigma_x \sigma_p = \hbar \left(n + \frac{1}{2}\right) \geq \frac{\hbar}{2}}$$

(b) $\varphi_1 = \left(\dfrac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{2}\, \xi\, e^{-\xi^2/2} \qquad \left(\xi \equiv \sqrt{\dfrac{m\omega}{\hbar}}\, x\right)$

$$\langle \hat{x}^2 \rangle = \int \varphi_1^* x^2 \varphi_1 \, dx =$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} 2 \int \left(\xi e^{-\xi^2/2}\right) \underbrace{\left(\frac{\hbar}{m\omega}\xi^2\right)}_{x^2} \left(\xi e^{-\xi^2/2}\right) \underbrace{\sqrt{\frac{\hbar}{m\omega}}\, d\xi}_{dx}$$

$$= \frac{2}{\sqrt{\pi}} \frac{\hbar}{m\omega} \underbrace{\int_{-\infty}^{\infty} \xi^4 e^{-\xi^2} d\xi}_{\frac{3}{4}\sqrt{\pi}}$$

$$\boxed{\langle \hat{x}^2 \rangle = \frac{3}{2} \frac{\hbar}{m\omega}}$$

same as in 1(a) for $n = 1$