<u>PHYSICS 847</u>

Home Work Assignment # 6 5/22/2009

<u>Due:</u> Tuesday, June 2, 2009

1. This problem focuses on the mathematics of **Gaussian Integrals**. It is worth doing carefully because these integrals are of central importance in statistical mechanics and quantum field theory. You will apply these techniques to the Ising model in Problem 2.

(a) Prove (once more!) the simple result that you have used all the time last quarter

$$\int_{-\infty}^{+\infty} dx \exp\left(-\frac{1}{2}Ax^2\right) = \left(\frac{2\pi}{A}\right)^{1/2}.$$

(b) Generalize this result to a multi-dimensional integral

$$\int D\mathbf{x} \exp\left(-\frac{1}{2}x_i A_{i,j} x_j\right) = \frac{(2\pi)^{N/2}}{\sqrt{\det\mathbf{A}}},$$

where $\mathbf{x}$ is an N-dimensional vector, $\mathbf{A}$ is a real, symmetric $N \times N$ matrix with positive eigenvalues, $\int D\mathbf{x} \equiv \int_{-\infty}^{+\infty} \prod_{i=1}^{N} dx_i$. Here and below we use the Einstein summation convention: all repeated indices are implicitly summed over, so that $x_i A_{i,j} x_j = \mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x}$.
Hint: Make a suitable change of variables. Why do we need the restriction on the sign of the eigenvalues?

(c) Show that

$$\int_{-\infty}^{+\infty} dx \exp\left(-\frac{1}{2}Ax^2 + bx\right) = \left(\frac{2\pi}{A}\right)^{1/2} \exp\left(b^2/2A\right)$$

by "completing the square" in the exponent.

(d) Now generalize this result to

$$\int D\mathbf{x} \exp\left(-\frac{1}{2}x_i A_{i,j} x_j + b_i x_i\right) = \frac{(2\pi)^{N/2}}{\sqrt{\det\mathbf{A}}} \exp\left[\frac{1}{2} b_i (A^{-1})_{i,j} b_j\right]$$

where all the symbols are the same as in part (b), and $\mathbf{b}$ is an N-dimensional vector with $b_i x_i = \mathbf{b}^T \cdot \mathbf{x}$. Hint: "Complete the square" in the exponent using a transformation $y_i = x_i - M_{i,j} b_j$ with an appropriately chosen $M_{i,j}$.

(e) Define

$$\langle Q \rangle = Z^{-1} \int D\mathbf{x} Q[\mathbf{x}] \exp\left(-\frac{1}{2} x_i A_{i,j} x_j\right) \text{ with } Z = \int D\mathbf{x} \exp\left(-\frac{1}{2} x_i A_{i,j} x_j\right).$$

Show that

$$\langle x_k x_l \rangle = (A^{-1})_{k,l}.$$

Hint: introduce "external sources" or "fields" b_i which add $b_i x_i$ to the exponent in Z as in part (d). Now take derivatives of $Z[\mathbf{b}]$ with suitable b_j's, and finally set all b's to zero, to compute the correlation function of interest.

The next step – which I do *not* want you to do as part of your homework – would be to obtain more complicated correlations like $\langle x_i x_j x_k x_l \rangle$ as sums of products of matrix elements of A^{-1}. This would prove the celebrated Wick's theorem.

2. In this problem you will learn: (1) To rewrite the Ising model as a field theory, i.e., a functional integral over real fields defined at each point on a lattice; (2) To see the connection between mean field theory and a saddle-point approximation to the functional integral; and (3) To see the connection between the Ising model and its Landau-Ginzburg Free energy functional. Since the calculations will get messy, you should focus on getting a qualitative feel rather than details like factors of π.

Consider the Ising model with the Hamiltonian

$$\mathcal{H} = -J \sum_{(i,j)} S_i S_j - \sum_i h_i S_i$$

with near neighbor ferromagnetic interactions and a spatially dependent dependent field on a lattice with total number of sites $= N$.

(a) Show that you can write its partition function in the form

$$Z = \mathrm{Tr} \exp\left[\frac{1}{2} S_i J_{i,j} S_j + S_i h_i\right]$$

where the implicit sum is over *all* sites i and j of the lattice (hence the factor of 1/2). Tr represents a sum over all spin configurations with $\{S_i = \pm 1\}$ the matrix $J_{i,j} = \beta J > 0$ when i and j are nearest neighbors, and $J_{i,j} = 0$ otherwise, and $H_i = \beta h_i$. (Factors of β are absorbed into J and H to simplify later notation).
Note: From now on I will use the summation convention of Problem 1 unless stated otherwise.

(b) We want to use the identity used in Problem 1(d) to rewrite Z using $b_i = S_i$ and $(A^{-1})_{i,j} = J_{i,j}$. This trick is called the Hubbard-Stratonovich transformation. The exponent in Z is then *linear* in S_i and thus the Trace can be done trivially. This simplification comes at the cost of introducing an integration $\int_{-\infty}^{+\infty} dx_i$ at each site. The point is that integrals are much easier to deal with in analytical calculations compared to sums over discrete spins (± 1).

But before you proceed as suggested, note that there is a problem in using the result of 1(d): the eigenvalues of the matrix $J_{i,j}$ are *not* positive definite. How can you see this most easily? (If you have trouble seeing this in general, first work out the 3-site problem to gain some intuition).
To deal with this technical problem, define $K_{i,j} = K_0 \delta_{i,j} + J_{i,j}$ and argue that for K_0 sufficiently large the eigenvalues of $K_{i,j}$ *are* positive definite. Show that

$$Z = \text{Const.} \mathrm{Tr} \exp \left[\frac{1}{2} S_i K_{i,j} S_j + S_i h_i \right]$$

where the shift in energy due to K_0 leads to the multiplicative constant. Such constants are not relevant for most purposes and we will simply ignore them.

(c) Now implement the strategy outlined above to show that

$$Z = \int D\mathbf{x} \exp\left(-\mathcal{S}[x_i, H_i]\right)$$

where multiplicative factors in Z are omitted, and

$$\mathcal{S}[x_i, H_i] = \frac{1}{2} x_i (K^{-1})_{i,j} x_j - \sum_i \ln\left[\cosh(x_i + H_i)\right].$$

It is convenient to define a new variable $\psi_i = x_i + H_i$ in terms of which

$$\mathcal{S}[\psi_i, H_i] = \frac{1}{2} \psi_i (K^{-1})_{i,j} \psi_j - \sum_i \ln(\cosh \psi_i) + H_i (K^{-1})_{i,j} \psi_j$$

where we have dropped an additive $HK^{-1}H$ term which will also not be important.

This is the functional integral representation of the Ising model. The integration is over the ψ-fields, where at each site of the lattice there is a real number ψ_i.

(d) To get a feeling for physical meaning of the ψ's, lets calculate the some expectation values. From the Z given in 2(a) show that $\langle S_i \rangle = \partial \ln Z / \partial H_i$, and let us work for simplicity at zero external field ($H_i = 0$). Next evaluate the the derivative w.r.t. H_i in the functional integral representation of part (c) to obtain

$$\langle S_i \rangle = (K^{-1})_{i,j} \langle \psi_j \rangle$$

where the expectation value on the RHS is defined as

$$\langle Q \rangle = \frac{\int D\psi Q[\psi] \exp(-\mathcal{S})}{\int D\psi \exp(-\mathcal{S})}$$

with $H_i = 0$. This shows that ψ_i is closely related to the magnetization; if we define $\phi_i = (K^{-1})_{i,j}\psi_j$ then its expectation value is the magnetization. Similarly its easy to see (though you don't need to do it for the HW) that $\langle S_i S_j \rangle$ is related to $\langle \phi_i \phi_j \rangle$.

(e) Next let us see the connection between the saddle point approximation for the functional integral and mean field theory. Set $H_i = 0$ for simplicity. Find the equation for ψ_i which minimizes $\mathcal{S}$ and call its solution $\overline{\psi}_i$. In terms of the the variables ϕ_i defined in part (d) above, show that

$$\overline{\phi}_i = \tanh\left(K_{i,j}\overline{\phi}_j\right).$$

For a spatially homogeneous order parameter $\overline{\phi}_j = \overline{\phi}$, this should remind you of the mean field theory equation for the magnetization. Thus the mean field approximation is equivalent to saying that a single (saddle-point) "configuration" $\overline{\phi}_i$ dominates the functional integral and all fluctuations about it are dropped in this approximation.

However there is a problem with the result we have derived! It depends upon the arbitrary K_0 (introduced in 2(b)), while if we had made no approximations (and kept all K_0-dependent multiplicative constants) the functional integral should have been completely independent of K_0.

The simplest (but certainly not rigorous!) way out of this problem is to argue as follows. Recall that K_0 was introduced earlier to ensure convergence of integrals and final answers should not depend on it. Now that we have obtained a finite result, we simply set $K_0 = 0$ at the end. (In fact, many of the standard textbooks in the field do this right from the beginning paying no attention to questions of convergence).

(f) To see the connection with Ginzburg-Landau theory, expand the result of part (c) in small ψ to get

$$\mathcal{S}[\psi_i, H_i] = \frac{1}{2}\sum_{i\neq j}\psi_i(K^{-1})_{i,j}\psi_j - + \sum_i \left[\frac{1}{2}\left(K_{i,i}^{-1} - 1\right)\psi_i^2 + \frac{1}{12}\psi_i^4 + \dots\right]$$

(where we now show all summations explicitly). Although, I don't expect you to show this, I note that by going to Fourier space and expanding in slow variations, the first term can be written in terms of $|\nabla\psi|^2$.

3. Consider temperatures slightly above the critical point T_c so that $t = (T - T_c)/T_c \ll 1$. (An identical analysis will work below T_c as well). The susceptibility diverges like $\chi(T) = \chi_0|t|^{-\gamma}$ and the correlation function has the long-distance behavior $G(r) \sim \exp\left(-r/\xi(T)\right)/r^{d-2+\eta}$ where the correlation length $\xi(T) = \xi_0|t|^{-\nu}$.

(a) Show, using the Fluctuation-Dissipation theorem, that the divergence in susceptibility as $T \to T_c$ is directly related to the divergence of the correlation length. Thus show that the critical exponents γ, ν and η are not independent, but are related via

$$\gamma = (2 - \eta)\nu$$

Check that this identity is obeyed by mean field exponents and also by the exponents known from the exact solution of the 2D Ising model.