

PHYSICS 847

**Home Work Assignment # 4**

4/21/2009

Due: Tuesday, 4/28/2009

**1.** Read Sections 3.1 (Mean Field Theory of the Ising Model), 3.2 (Bragg-Williams Approximation) and 4.1 (Order-Disorder Transition) of your text book: Plischke and Bergersen, 3rd Edition.

Then do Problem 4.1 on page 137 which is based on the material in Sec. 4.1.

**2.** In class (and in the text book) mean field theory was presented as a approximate solution of a given problem, say the nearest neighbor Ising model. Here we will view mean field theory as the exact solution to a problem which is related to the original (short-range) problem.

Our goal is to see that the mean field result is the *exact* solution of the “infinite range” Ising model in which each spin interacts with *every other* spin. This model consists of spins  $S_i = \pm 1$  ( $i = 1, \dots, N$ ) with the Hamiltonian:

$$\mathcal{H} = \frac{-J_0}{2} \sum_{i \neq j} S_i S_j - H \sum_i S_i.$$

Note that the sum in the interaction term includes *all* pairs of spins (in marked contrast with the local Hamiltonians discussed in class).

(a) Explain *why* this model has sensible thermodynamic properties *only if* we take  $J_0 = J/N$ , with  $J$  of order unity, in the thermodynamic limit  $N \rightarrow \infty$ .

(b) Prove the mathematical identity

$$\exp(a^2) = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2 + \sqrt{2}ax\right).$$

(c) Rewrite the interaction term in  $\mathcal{H}$  in terms of  $(\sum_i S_i)^2$ . Then make use of the result of part (b) to write the Boltzmann factor  $\exp(-\beta\mathcal{H})$  as an integral over an auxiliary variable  $x$  of a function which is an exponential whose exponent depends only *linearly* on  $(\sum_i S_i)$ .

(d) Show that the Trace in  $\text{Tr} \exp(-\beta\mathcal{H})$ , i.e., the sum over all spin configurations, can now be done trivially. After a suitable rescaling of  $x$  to

$y = \text{const.} \times x$ , where you should determine the constants here and below, show that the partition function of the infinite-range Ising model can be written as

$$Z = \text{Const.} \times \int_{-\infty}^{+\infty} dy \exp[-N\beta L(y)]$$

where

$$L(y) = Jy^2/2 - \beta^{-1} \ln [2 \cosh(\beta Jy + \beta H)].$$

(e) The integral in part (d) can be evaluated by the steepest descent or saddle point method in the large  $N$  limit.

Please review this method from your favorite mathematical physics book and give the reference to the book you studied. [I like G. Arfken and H. Weber, “Mathematical Methods for Physicists” (5th Ed.), (Academic Press, 2001) but any other book should suffice].

Using this method show that you obtain, as  $N \rightarrow \infty$ ,

$$Z = \exp[-N\beta L(y^*)]$$

with  $y^*$  determined by the solution of  $dL(y^*)/dy = 0$ .

(f) Find the equation for  $y^*$  for arbitrary  $T$  and  $H$  and solve it graphically for the case of  $H = 0$  to determine the global minimum of the Free energy density  $F(T)/N$ .

(g) Compute the magnetization  $m$  of this Ising model using the standard definition in terms of the derivative of the Free energy with respect to field, and show that  $m = y^*$ .

Note that the exact magnetization and free energy results obtained for the infinite-range Ising model are in one-to-one correspondence with the mean field approximation results for the nearest neighbor Ising model.