

PHYSICS 847

Home Work Assignment # 3

4/14/2009

Due: Tuesday, April 21

1. The equation of state for a van der Waals (VdW) fluid is given by

$$\left(p + \frac{a}{v^2}\right)(v - b) = k_B T.$$

p is the pressure, T the temperature and $v = V/N$ the volume per particle.

(a) Show that the critical point corresponds to

$$p_c = \frac{a}{27b^2}; \quad k_B T_c = \frac{8a}{27b}; \quad \text{and} \quad v_c = 3b.$$

(b) Show that the VdW theory implies that *all* fluids should have a “universal” equation of state when described in terms of the dimensionless variables $\tilde{p} = p/p_c$, $\tilde{t} = T/T_c$, and $\tilde{v} = v/v_c$.

2. Compare the *irreversible* free expansion at a constant T for an ideal classical gas with that of a real classical gas ($n\lambda_T^3 \gg 1$), modeled with the van der Waals equation of state with $T > T_c$. Comment on the differences that arise as a result of inter-particle interactions.

3. We want to understand the properties of the van der Waals fluid as one approaches the critical point $T \rightarrow T_c$. Using $\delta p = (p - p_c)/p_c$, $\delta v = (v - v_c)/v_c$ and $\delta t = (T - T_c)/T_c$ to describe deviations from the critical point, show that the isothermal compressibility $K_T = -V^{-1}(\partial V/\partial p)_T$ *diverges* along the critical isochore $\delta v = 0$ as $K_T \sim |\delta t|^{-\gamma}$ for $\delta t \rightarrow 0$ from above. What is the value of the critical exponent γ ?

4. Consider a classical gas composed of N particles of mass m occupying a box of volume V at temperature T . The particles interact pairwise via an interparticle potential, so that the Hamiltonian is given by

$$H = \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{i,j} V(|\mathbf{r}_i - \mathbf{r}_j|)$$

Let the potential be of the form

$$V(r) = \frac{A}{r^\alpha}$$

where A and α are positive quantities.

We would like to ensure that the system actually possesses a sensible thermodynamic limit so that, for instance, the internal energy is an extensive quantity. This requires that the potential V decay sufficiently rapidly. *Find the condition* on α so that the thermodynamic limit exists.

Hint: Using the density “operator” $n(\mathbf{r}) = \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i)$, show that the interaction energy can be written in the form

$$\int d\mathbf{r} \int d\mathbf{r}' n(\mathbf{r}) V(|\mathbf{r} - \mathbf{r}'|) n(\mathbf{r}').$$

Next, make the approximation of a *constant* density in the fluid and determine the convergence of integrals.

5. At low temperatures and atmospheric pressure ${}^3\text{He}$ is a liquid, but can be turned into a solid by the application of pressure. In this problem we will be interested in the behavior of the phase boundary between liquid and solid phases as a function of temperature and pressure over a range of temperatures below 0.3 K.

(a) In the solid phase the He atoms form a crystal lattice. Each ${}^3\text{He}$ atom has a nuclear spin of $1/2$. Assuming that in this range of temperature the nuclear spins are effectively noninteracting, find the entropy of the system per atom s_S arising from the spins. This is the dominant part of the entropy of the solid.

(b) In this temperature range the liquid may be crudely modeled as an ideal Fermi gas. If the specific volume of the liquid is 46\AA^3 per atom, find the degeneracy temperature T_F ($= \epsilon_F/k_B$) and an expression for the entropy per atom of the system s_L at temperatures well below the degeneracy temperature.

(c) Using the Clausius-Clapeyron equation find the slope of the melting curve (dP/dT) in the limit of $T \ll T_F$ and comment on the physical significance of its sign. You are given that in the solid, each atom occupies an effective volume of 43\AA^3 .