

PHYSICS 847

Home Work Assignment # 2

4/7/2009

Due: Please note: Both HW # 1 and # 2 are due on Tues., Apr. 14, 2009

1. Bose-Einstein Condensation (BEC) of a dilute gas of alkali atoms is achieved in the laboratory by confining bosons of mass m in a harmonic (magnetic or magneto-optical) trapping potential. Ignoring inter-particle interactions, the N -particle Hamiltonian is simply the sum of N single particle Hamiltonians each of the form

$$H = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}m (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$$

where $\omega_x, \omega_y, \omega_z$ are the trap frequencies in an anisotropic trap and $\bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$ is their geometric mean.

(a) For $\epsilon \gg \bar{\omega}$ the discrete energy eigenvalues may be approximated as a continuous distribution with the density of states $g(\epsilon)$. Show that the DOS in the anisotropic trap is given by

$$g(\epsilon) = \frac{\epsilon^2}{2(\hbar\bar{\omega})^3}.$$

(This result is identical to what you did last quarter in when you worked out a problem on a Fermi gas in a harmonic trap.)

(b) Find the BEC temperature T_c by writing N as a sum over states in the trap and showing that for $T < T_c$ there must be macroscopic occupation of the ground state. How does your answer differ from the BEC temperature in a uniform ideal Bose gas discussed in class? Please explicitly evaluate all the integrals here and below in terms of the Gamma function and the Reimann zeta function $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$, where $\zeta(2) = \pi^2/6$; $\zeta(3) \simeq 1.202$; $\zeta(4) = \pi^4/90$.

(c) Give a simple argument to understand the scaling of T_c with N obtained in part (b) using the criterion of interparticle spacing comparable to thermal deBroglie wavelength. Hint: the characteristic length ℓ in computing the “number density” in a trap is given by setting $m\bar{\omega}\ell^2$ of order the thermal energy.

(d) Find the internal energy of the system at $T = T_c$. How does $U(T_c)$ compare with for $\hbar\bar{\omega}$ for $N \gg 1$.

(e) Find the condensate fraction for $T < T_c$.

(f) Consider $N = 10^7$ atoms of ^{87}Rb in a trap with frequencies $(\omega_x, \omega_y, \omega_z) = 2\pi(250, 670, 7)$ rad/s. What is the BEC temperature T_c in K?

2. In this problem we continue our study of the trapped Bose gas of the previous Problem. Here we will learn how the density profile $n(\mathbf{r})$ and the momentum distribution $n_{\mathbf{k}}$ of this system look completely different in the low temperature BEC state and in the high temperature classical regime. It is precisely through imaging experiments, which probe $n(\mathbf{r})$ and $n_{\mathbf{k}}$, that BEC was first studied in the 1995 Nobel Prize winning experiments of Cornell and Weiman and of Ketterle. In the actual experiments, weak repulsive interactions between the bosons alter the (non-interacting) results derived below, but you can nevertheless get a flavor of what is involved.

(a) At $T = 0$ all of the bosons are in the single-particle ground state of the harmonic trap described by the normalized wavefunction

$$\phi_0(\mathbf{r}) = C \exp\left(-\frac{x^2}{2a_x^2} - \frac{y^2}{2a_y^2} - \frac{z^2}{2a_z^2}\right).$$

Find the normalization C and express the “widths” in the three directions a_i in terms of m and ω_i .

(b) Argue that the density profile of the gas at $T = 0$ is given by

$$n(\mathbf{r}) = N|\phi_0(\mathbf{r})|^2$$

(c) Let $\Phi_0(\mathbf{k})$ be the Fourier transform of the ground state wavefunction $\phi_0(\mathbf{r})$. Show that the $T = 0$ momentum distribution is given by

$$n_{\mathbf{k}} = N|\Phi_0(\mathbf{k})|^2$$

, and argue that it looks like a Maxwellian distribution with different “temperatures” T_i for the three directions.

(d) Next consider the classical regime $T \gg T_c$. Show that the density profile is then given by

$$n(\mathbf{r}) = \frac{N}{\pi^{3/2} R_x R_y R_z} \exp\left(-\frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} - \frac{z^2}{R_z^2}\right).$$

How are the R_i 's related to the temperature?

(e) Compare the density profiles at $T = 0$ [part (a)] and $T \gg T_c$ [part (d)] and argue that the density profile of the “thermal cloud” at high temperatures is much broader than the narrow condensate peak at $T = 0$.

(f) Next calculate the momentum distribution $n_{\mathbf{k}}$ in the classical regime for $T \gg T_c$. Show that it is isotropic Maxwellian distribution in \mathbf{k} -space whose width is determined by temperature.

(g) Finally, compare the momentum distributions at $T = 0$ [part (b)] and $T \gg T_c$ [part (f)]. Show that the isotropic \mathbf{k} -space width of the “thermal cloud” is much broader than the anisotropic momentum-space widths of the condensate.