PHYSICS 847

Home Work Assignment # 1 4/2/2009

Due: Thurs., Apr. 9, 2009

1. The goal of this problem is to help you understand in detail *why* it suffices to explicitly keep track of just one state, corresponding to $\mathbf{k} = 0$, and account for all other states using the DOS, in the study of Bose-Einstein Condensation (BEC).
Consider N non-interacting bosons of mass m in a large box of dimensions $L \times L \times L = V$ with rigid walls. For simplicity, consider periodic boundary conditions. In addition to the single-particle ground state with $\mathbf{k} = 0$, let us also explicitly keep track of the excited state with the lowest non-zero energy, say ϵ_1 corresponding to momentum $\mathbf{k_1}$. (Actually this state is triply degenerate, but lets ignore this since it does not affect the arguments that follow).
(a) Show that $\epsilon_1 \sim \mathcal{O}\left(V^{-2/3}\right)$ and the the chemical potential for $T < T_c$ is negative with $|\mu| \sim \mathcal{O}\left(V^{-1}\right)$. Using these results, show that the number density of bosons in the state $\mathbf{k_1}$ in the condensed phase is of $\mathcal{O}\left(V^{-1/3}\right)$, as opposed to the number density in the $\mathbf{k} = 0$ condensate which is $\mathcal{O}(1)$. Thus argue that one does not need to individually keep track of any states other than $\mathbf{k} = 0$.
(b) To get a better feel for the argument of part (a), make the following numerical estimates for a system of 10^{22} ^{4}He atoms, *assumed to be noninteracting*, in a cubic container of side L = 1 cm. Assuming the wavefunctions correspond to periodic boundary conditions, estimate ϵ_1 in degrees K. Next estimate the chemical potential μ (in degrees K) at a temperature of 1 mK. Compute the average occupancy of the $\mathbf{k} = 0$ and $\mathbf{k_1}$ states, and show that the fraction of bosons in the $\mathbf{k_1}$ state is negligible compared to that in the $\mathbf{k} = 0$ state.

2. Consider a uniform gas of N non-interacting bosons in a d-dimensional "volume" L^d with an energy dispersion $\epsilon(p) = Ap^s$. (We had considered the case of s = 2 and d = 3 in class.) Find the condition on s and d for Bose-Einstein condensation (BEC) to occur at a finite temperature. Does BEC occur for non-relativistic bosons ($s = 2$) in $d = 2$ dimensions?

3 & 4 Two problems on dimensional analysis and scaling (see other side).

Exercise 1–1

Dimensional analysis (DA) is often a powerful tool in physics. This question requires you to use dimensional arguments to solve a couple of interesting problems. DA is usually used in two ways: (1) The fundamental theorem of DA asserts that in any physical problem involving a number of dimensionful quantities, the relationship between them can be expressed by forming all possible independent dimensionless quantities, denoted by $\Pi, \Pi_1, \Pi_2, \ldots, \Pi_n$. Then the solution to the physical problem is of the form $\Pi = f(\Pi_1 \ldots \Pi_n)$, where f is a function of n variables. (2) Sometimes there is only one dimensionless combination of variables relevant to a given problem. Then (1) implies $\Pi =$ constant.

(a) By noting that the area of a right-angled triangle can be expressed in terms of the hypotenuse and (*e.g.*) the smaller of the acute angles, prove Pythagoras' theorem using dimensional analysis. You will find it helpful to construct a well-chosen line in the right-angled triangle. *Note: the whole point of dimensional analysis is that you do NOT need to solve for the functional form of the solution to a given problem. Thus, in this question, you must pretend that you do not know trigonometry.*

(b) Now consider the case of Riemannian or Lobachevskian geometry (*i.e.* the triangle is drawn on a curved surface such as a riding saddle or a football). What happens in this case?

Exercise 1–2

In 1947, a sequence of photographs of the first atomic bomb explosion in New Mexico in 1945 were published in *Life* magazine. The photographs show the expansion of the shock wave caused by the blast at successive times in ms. From the photographs, one can read off the radius of the shock wave as a function of time: the result is shown in the accompanying table. Assuming that the motion of the shock is unaffected by the presence of the ground, and that the motion is determined only by the energy released in the blast E and the density of the undisturbed air into which the shock is propagating, ρ, derive a scaling law for the radius of the fireball as a function of time. Use the data from the photographs to test your scaling law and hence deduce the yield of the blast. *You must test your scaling law by plotting a graph. You should consider carefully and then explain what is the most useful graph to plot.* You should assume that all numerical factors are of order unity. Although the photographs were declassified in 1947, the yield of the explosion was to remain classified until several years later.

Table 1.1 RADIUS R OF BLAST WAVE AFTER TIME T

T/msec	R/m
0.10	11.1
0.24	19.9
0.38	25.4
0.52	28.8
0.66	31.9
0.80	34.2
0.94	36.3
1.08	38.9
1.22	41.0
1.36	42.8
1.50	44.4
1.65	46.0
1.79	46.9
1.93	48.7
3.26	59.0
3.53	61.1
3.80	62.9
4.07	64.3
4.34	65.6
4.61	67.3
15.0	106.5
25.0	130.0
34.0	145.0
53.0	175.0
62.0	185.0