

PHYSICS 846

Home Work Assignment # 7

3/3/2009

Due: Tues., Mar. 10, 2009.

Note: Final Exam on Friday, Mar. 13, 4:30 – 6:30 PM, Smith 1042.

1. In class we derived properties of blackbody radiation at a temperature T by considering the statistical mechanics of quantum harmonic oscillators describing the modes of e.m. radiation in a cavity of volume V . The number of these oscillators was infinite (no ultraviolet cutoff) even for a fixed volume V , and we analyzed the problem in the Canonical Ensemble.

Here we wish to rederive the same results, but by looking at a somewhat different set of degrees of freedom: the *excitations* in these e.m. modes which constitute a gas of noninteracting bosonic particles, the *photons*. The number of photons is not fixed, but determined by T and V (as we already saw in class and in Problem 6 of HW# 4); we are thus forced to work in the Grand Canonical Ensemble. The minimum condition of the Free energy implies $(\partial F/\partial N)_{T,V} = 0$, so that the chemical potential for photons $\mu = 0$.
(a) Start with the general result derived in class for bosons

$$\mathcal{Z} = \prod_i \left[\frac{1}{1 - e^{-\beta(\epsilon_i - \mu)}} \right]$$

and specialize to the photon gas whose states are labeled by a wave vector \mathbf{k} and two transverse polarizations, with energy $\epsilon(\mathbf{k}) = c\hbar|\mathbf{k}|$, where c is the speed of light. Find $\mathcal{Z}(T, V, \mu = 0)$.

Hence determine the following quantities for the photon gas: (b) the number density $n(T)$ (c) the internal energy density $U(T)/V$; (d) the entropy density $S(T)/V$; and (e) the equation of state $p(T, V)$. What is the relationship between p and U/V ?

2. Estimate the Fermi energies E_F (in K) for (a) liquid ${}^3\text{He}$ and (b) nuclear matter and compare the numbers with E_F for electrons in metals.

For case (a) note that the ${}^3\text{He}$ atom is a spin $I = 1/2$ fermion and the density of the liquid is $\simeq 0.081 \text{ g cm}^{-3}$.

For case (b) use the empirical relation $R \simeq 1.3 \times 10^{-13} \text{ cm} \times A^{1/3}$ for the radius R of a nucleus containing A nucleons. For simplicity assume that there are equal numbers of protons and neutrons and find the E_F for either

of them. Note it is meaningless to talk of E_f for “nucleons” since protons and neutrons are distinguishable particles. Is it sensible to use a non-relativistic approximation for nuclear matter?

3. Graphene: Solve Kardar, Problem 17 of Chapter 7 on page 210.

Hints and Comments: In part (b), A is the area of the graphene sheet. In part (c) you should reduce the answer to a form that involves the product of all quantities of interest with a dimensionless integral of dimensionless variables. You do not have to evaluate this integral. In part (e), you should do better than a “qualitative” estimate of the phonon specific heat. Given the speed of sound make a semi-quantitative estimate of the lattice contribution and compare it with the electronic one.

4. Consider an AMO (atomic-molecular-optical physics) experiment with gas of N non-interacting spinless Fermions of mass m trapped in a harmonic potential

$$V(x, y, z) = \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2).$$

Here $\omega_x, \omega_y, \omega_z$ are the trap frequencies in an anisotropic trap and their geometric mean is $\bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$. In this question we look at various aspects of the low temperature thermodynamics of the N -Fermion system.

(a) For $\epsilon \gg \bar{\omega}$ the discrete energy eigenvalues may be approximated as a continuous distribution with the density of states $g(\epsilon)$. Show that the DOS in the anisotropic trap is given by

$$g(\epsilon) = \frac{\epsilon^2}{2(\hbar\bar{\omega})^3}.$$

Note that here $g(\epsilon)d\epsilon$ is the number of states in the energy interval $(\epsilon, \epsilon + d\epsilon)$ (without any volume factor).

(b) Using the Sommerfeld expansion, determine the internal energy U and the number N as a function of μ and T . Assume that, at the low temperatures of interest, the chemical potential $\mu \gg \hbar\bar{\omega}$. For both U and N retain the leading temperature-dependent correction to the $T = 0$ result.

(c) Use the results of part (b) to show that:

$$\mu = A\hbar\bar{\omega}N^{1/3}(1 + ax + \dots) \quad \text{and} \quad U = B\hbar\bar{\omega}N^{4/3}(1 + bx + \dots)$$

where $x = (k_B T / \hbar\bar{\omega})^2 N^{-2/3}$, and A, B, a, b are dimensionless constants that you should determine.