PHYSICS 846

Home Work Assignment # 5 (corrected version) 2/18/2009

Due: **Tues., Feb. 24, 2009**.

1. *Dilute polyatomic gases:* Work through Sec. 6.1 of Kardar (p. 156 - 161).

2. Consider the lattice vibrations of a solid in d spatial dimensions. Further assume that the phonons in this solid have a dispersion $\omega(k) = Ak^s$ with an arbitrary $s > 0$.
(a) Show that the low temperature specific heat then depends on temperature as $C(T) = BT^a$ and determine the exponent a in terms of d and s.
(b) The experimentally measured specific heat of graphite over a certain range of temperature behaves as $C \propto T^2$. Suggest a plausible explanation for this behavior.

3. Staring with definition of the chemical potential as a derivative of the Helmholtz free energy: $\mu(T, V, N) = (\partial F/\partial N)_{T,V}$, show that it can also be written as a derivative of the entropy or internal energy.

$$\mu = -T\left(\frac{\partial S}{\partial N}\right)_{U,V} \quad \text{and} \quad \mu = \left(\frac{\partial U}{\partial N}\right)_{S,V}$$

4. The isothermal compressibility is defined as $\kappa_T = -\frac{1}{V}\left(\partial V/\partial p\right)_{N,T}$. Show that we can also write it in the very useful form

$$\kappa_T = \frac{1}{n^2}\left(\frac{\partial n}{\partial \mu}\right)_T$$

where $n = N/V$ is the number density.
Hint: Argue, using the extensivity of the Helmholtz free energy, that $F = Vf(n, T)$ where f is the free energy density, an intensive function of intensive variables. Then calculate μ, p, $1/\kappa_T$ etc. as derivatives of F.

5. (a) Show that the partition function of a classical ideal gas of N particles in the canonical ensemble at fixed T and V is given by

$$Z_N(N, T, V) = \frac{1}{N!}\left(\frac{V}{\lambda_T^3}\right)^N,$$

where λ_T is the thermal deBroglie wavelength.
(b) Using Z_N from part (a), show that the grand canonical partition function (or Gibbs sum) for the same gas at given μ, T and V is given by

$$\mathcal{Z}(\mu, T, V) = \exp\left(\frac{yV}{\lambda_T^3}\right)$$

where $y = \exp(\beta\mu)$ is called the *fugacity* (often denoted by z, but we'll call it y as there are too many z's).
(c) Show that the probability there are N atoms in volume V of gas in diffusive contact with a reservoir is given by the Poisson distribution

$$P(N) = \alpha^N \exp\left(-\alpha\right)/N!$$

with $\alpha = yV/\lambda_T^3$. Check that this distribution is properly normalized and show that $\alpha = \langle N \rangle$ the average number.

6. Prove that number fluctuations in the grand canonical ensemble are small in the thermodynamic limit. Toward this end, derive the following results, where the symbols have their standard meaning: $\mathcal{Z}$ is the grand canonical partition function, $\beta = 1/k_B T$, and the partial derivatives with respect to the chemical potential μ are taken with the temperature T and volume V held fixed. The angular brackets $\langle \ldots \rangle$ denote a grand canonical average.
(a)

$$\langle N \rangle = \frac{1}{\beta}\frac{\partial}{\partial\mu}\log\mathcal{Z}$$

(b)

$$\langle N^2 \rangle = \frac{1}{\beta^2}\frac{1}{\mathcal{Z}}\frac{\partial^2\mathcal{Z}}{\partial\mu^2}$$

(c)

$$\langle (\delta N)^2 \rangle \equiv \langle (N - \langle N \rangle)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2$$

(d)

$$\langle (\delta N)^2 \rangle = \frac{1}{\beta}\frac{\partial \langle N \rangle}{\partial\mu}$$

(e)

$$\frac{\langle (\delta N)^2 \rangle}{\langle N \rangle^2} = \frac{n\kappa_T}{\beta N}$$

which in the limit of large volume vanishes (provided the compressibility is finite). What is the result of part (e) for the classical, ideal gas?