PHYSICS 846

Home Work Assignment # 3 1/27/2009

Due: **Tues., Feb. 3, 2009**.

Reminder: The **Mid Term Exam** is on Tuesday, Feb. 10, 2009.

1. Consider a gas of ^{4}He atoms in a container of volume 0.1 liter at 300 K and 1 atm pressure.

(a) Show that the system can be adequately described as a classical, ideal gas. You need to justify why one can ignore (i) interatomic interactions *and* (ii) quantum statistics for this gas of particles.

We want to make a rough estimate for the time scale for a very large, highly improbable fluctuation in which the system is in a configuration where all the atoms are in one-half, say the right half, of the container.

(b) First, estimate the number of micro-states accessible to the system in its initial configuration.

(c) Next, let the gas be isothermally and reversibly compressed to a volume of 0.05 liter. Find the number of micro-states accessible now. Thus estimate the ratio:

$$\frac{\text{\# of states for which atoms are in the right half}}{\text{\# of states for which atoms are anywhere in the container}}$$

(d) Let the rate of collision for an atom be $\simeq 10^{10}$ s^{-1}. Estimate the total number of collisions for all the atoms in the system in one year, and use this as a very crude measure of the frequency with which the "state of the system changes". (Note that the final answer is not affected by the crudeness of this estimate!)

Estimate the number of years you expect to wait before all the atoms are in one-half of the volume starting from an equilibrium configuration. How does this compare with the age of the universe?

2. Matter and radiation are present in the sun at very high temperatures. We would like to determine which of these dominates the pressure and stabilizes the sun against its own gravity.

Assume that the temperature of the sun is $T = 1.6 \times 10^7$ K, the total density of hydrogen gas is $\rho_H = 6 \times 10^4 \text{kg m}^{-3}$ and the density of helium is $\rho_{He} = 1 \times 10^5$ kg m^{-3}.

(a) Argue that at these temperatures the atoms are ionized, but the He-nuclei (alpha particles) do not break up into protons and neutrons. Hence we have a soup of electrons, protons, alpha particles and photons.

(b) Calculate the thermal deBroglie wavelengths of the matter particles at these temperatures. Assume that you can ignore the interactions between the charged particles in this plasma and show that you are justified in treating each species of matter as a classical gas. For which particles is this assumption the least justified?

(c) Calculate the pressure due to the matter particles. (Use GPa, or Giga-Pascals, as the unit of pressure here and below.)

(d) Calculate the pressure due to radiation using the formula

$$p = \frac{\pi^2 k_B^4}{15\hbar^3 c^3} T^4.$$

(We will derive this formula in class very soon). Which is the dominant component – radiation or matter – in stabilizing the sun against its own gravity?

3.Two-Level Systems: Read Section 4.3 (Microcanonical ensemble) and Section 4.7 (Canonical ensemble) of your text book on the statistical mechanics of two-level systems. You do not need to write up your "solution" but be sure to work through the algebra and understand it in detail.

4. A set of N electrical dipoles each with a dipole moment μ are on the sites of a lattice occupying a volume V. For simplicity we will assume that the dipoles are classical, non-interacting objects whose kinetic energy can be ignored. The energy of the system in an external electrical field $\mathbf{E}$ is given by summing $-\mu \cdot \mathbf{E}$ for each dipole. (Note: in this problem E denotes an electric field, *not* energy.)

(a) Working in the **canonical ensemble**, find the partition function Z for the system of N dipoles. Choosing the z-axis along the external $\mathbf{E}$ field, we may describe a microstate by specifying θ_i and ϕ_i for each dipole, $i = 1, \ldots, N$. Note that the dipoles are distinguishable since they occupy the sites of a lattice and can be labeled.

(b) Calculate the electrical polarization $\mathbf{P}(T, \mathbf{E})$ of the system at temperature T, where The polarization of the system is defined as the net dipole moment per unit volume.

(c) Find the dielectric susceptibility of the system

$$\chi(T) = \frac{1}{\epsilon_0} \left(\frac{\partial P}{\partial E} \right)_T$$

in the weak field limit $E \to 0$.

(d) Show that the root mean square fluctuation of the polarization P in the absence of an electric field can be simply related to the susceptibility χ.

(e) Calculate the dielectric constant $\epsilon = 1 + \chi$ for ice at $T = 0°$ C assuming (as above) that the dipole moments of the water molecules do not interact with one another (which is a somewhat drastic and unrealistic approximation). You are given that the dipole moment of a single H_2O molecule is $\mu = 6.2 \times 10^{-30}$ Coulomb-meter, the density of ice is 0.9×10^3 kg/m^3 and the permittivity of free space is $\epsilon_0 = 8.85 \times 10^{-12}$ in SI units.

5. In this problem we will return to the *classical ideal gas in a harmonic trap* that we treated in the microcanonical ensemble in Problem 4 of HW#3. Here we will treat the same problem in the **canonical ensemble**.

(a) Show that for a noninteracting system of N-particles the Hamiltonian breaks up into a sum

$$\mathcal{H} = \sum_{i=1}^{N} H\left(\mathbf{p}_i, \mathbf{r}_i\right); \quad \text{where} \quad H\left(\mathbf{p}, \mathbf{r}\right) = \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2 \mathbf{r}^2}{2}.$$

Then show that the N-particle partition function Z_N at a temperature $k_B T = \beta^{-1}$ can be written as

$$Z_N = \left(Z_1\right)^N / N!$$

where Z_1 is a six dimensional integral defining the partition function of a single classical particle.

(b) Prove the following definite integrals which will be needed in the subsequent analysis.

$$\int_{-\infty}^{+\infty} dx \exp\left(-\alpha x^2\right) = \left(\frac{\pi}{\alpha}\right)^{1/2},$$

$$\int_{-\infty}^{+\infty} dx\ x^2 \exp\left(-\alpha x^2\right) = \frac{\sqrt{\pi}}{2\alpha^{3/2}}.$$

(c) Compute the integrals required to find an explicit result for Z_N as a function of N and T, and determine the Helmholtz Free energy F for the N-particle system.

(d) Using your result for F, find the entropy S and the internal energy U.

(e) Show that the result for U can simply be understood in terms of each of the $6N$ "degrees of freedom" contributing $k_B T/2$.
Hint: Show – by explicit calculation – that each Cartesian component $p_{i,a}$ and $r_{i,a}$ with $i = 1, \ldots, N$ and $a = x, y, z$, which counts as a separate degree of freedom, contributes

$$\frac{\langle p_{i,a}^2 \rangle}{2m} = \frac{k_B T}{2} \quad \text{and} \quad \frac{m\omega^2 \langle r_{i,a}^2 \rangle}{2} = \frac{k_B T}{2},$$

where $\langle \ldots \rangle$ denotes a thermal average.